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DEMONSTRATION IN GEOMETRY: HISTORICAL AND PHILOSOPHICAL PERSPECTIVES

DEMONSTRAÇÃO EM GEOMETRIA: PERSPECTIVAS HISTÓRICA E FILOSÓFICA

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Abstract: In this article, we weave historical-philosophical reflections about demonstration in mathematics, based on works of researchers that discuss the different philosophical perspectives on the topic, more specifically on geometry. We focus first on demonstration and its relationship with intuition and figural representations. Second, we criticize Poincaré's conception of mathematical demonstration. Third, we reflect, in a non-exhaustive way, on the philosophy of demonstration in geometry, confronting Kant's conceptions with the axiomatizations of the non-Euclidean geometries. In this text, we do not adopt a single definition that would cover all modes of scientific validation, since we admit the possibility of an evolution of ideas about the validity of a proposition. Not to fall into the symmetrical flaws of the glorification of the Ancients or even being ungrateful to them, we must start from the naive idea that the demonstration has a historical origin and, therefore, maintains a historical character, but we should be more attentive to what characterizes, in its particularity or even its uniqueness, the productions of past and present centuries.

Keywords: Philosophy of demonstration; Axiomatization; Induction; Intuition; Representation.

Resumo: Tecemos neste artigo reflexões histórico-filosóficas da demonstração em matemática, apoiando-se em trabalhos de pesquisadores em que discutem diferentes perspectivas filosóficas da demonstração em matemática, mais especificamente em geometria. Focamos, em primeiro lugar a demonstração e sua relação com a intuição e as representações figurais. Em um segundo momento, apresentamos uma crítica sobre a concepção de Poincaré sobre a demonstração matemática. Na terceira parte, tecemos reflexões, de modo não exaustiva, sobre a filosofia da demonstração em geometria, confrontando as concepções Kant às axiomatizações das geometrias não-euclidianas. Neste texto, não adotamos uma definição única que cobriria todos os modos de validação cientifica, pois admitimos a possibilidade de uma evolução de ideias sobre a validade de uma proposição. Para não cair nas falhas simétricas da glorificação dos Antigos ou mesmo na ingratidão em relação a eles, devemos partir da ideia não ingênua de que a demonstração tem uma origem histórica e que, portanto, mantém um caráter histórico, mas deveríamos estar mais atentos ao que caracteriza em sua particularidade ou até sua singularidade, as produções dos séculos passados e presentes.

Palavras-chave: Filosofia da demonstração; Axiomatização; Indução; Intuição, Representação.

1 Introduction

Demonstration occupies a central place in mathematics, as it is the method of proof whose systematic use characterizes this discipline among the sciences. We usually locate the roots of mathematical demonstration in Classical Antiquity, precisely in Greece

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in the sixth century BC. By placing the appearance of the demonstration among Greeks, we are not neglecting the existence of proofs, justifications or verifications of different levels, for example, in Egypt, where the accuracy of calculations made by scribes was generally proven by verifying the result.

The demonstration among the Greeks is a consequence of reflective thinking influenced by the political-social and philosophical demands that were established by the need to "convince" the other. The triad Socrates, Plato, and Aristotle had the function of supplanting, through reflective thinking, the primitive mythical belief. Human activities in the "πολις" (polis = city) reached their high point of expression. Reflective thinking moved from the cosmological to the anthropological concern with the sophists - masters of rhetoric and eloquence in democratic Athens, who needed to prepare citizens to run for public office through free elections.

A reflection on the demonstration can take different paths. A psychological investigation is certainly possible. Demonstrative speech is demanding, scrupulous, unlike ordinary speech, the vehicle of our conventional ideas. Certainly, rigor honours those whose existence is based on principles. But questioning this rigor is not useless, as it leads us to distinguish control or constancy of mania or obsession, a kind of rigidity taken to the extreme, to the point of becoming illusory (GOMBAUD, 2007). An epistemological and philosophical investigation is also possible. In this article, we chose to weave historical-philosophical reflections about the demonstration in mathematics.

These reflections are supported by the work of researchers that discuss different philosophical perspectives of the demonstration in mathematics, more specifically in geometry. In the first part of this text, we discuss the demonstration and its relationship with intuition and figural representations. We rely mainly on the reflections by Bonnay and Dubucs (2011), Rouche (1989), and Duval (2005), among others. In the second part, we present Globot's (1907) criticism of Poincaré's theory about his concept of mathematical demonstration. With arguments with which we agree, this author asserts, among other aspects, that it is false to think that any reasoning that proceeds "from the private to the general" is inductive, as Poincaré thinks. In the third stage, we reflect, in a non-exhaustive way, on the philosophy of demonstration in geometry, confronting Kant's conceptions with the axiomatizations of the non-Euclidean geometries. We rely mainly on Chauve's (2006) work.

In this text, we have not adopted a single definition that would cover all modes of scientific validation. Because it is important to admit that ideas on the accuracy of a

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statement or, as the logician would say, on the validity of a proposition, may evolve (GOMBAUD, 2007). We agree with the author when he asserts that -not to fall into the symmetrical flaws of the glorification of the Ancients or even being ungrateful to them- we must start from the non-naive idea that the demonstration has a historical origin; therefore, it maintains a historical character. If we support the hypothesis that the representation of the world has changed considerably, as well as economic and political conditions, it becomes almost undeniable that Greek scholars have not demonstrated their propositions in the same way that professional mathematicians do today. Thus, we should carefully consider what characterizes the productions of past centuries in their particularity or even their uniqueness.

2 Demonstration: the role of intuition and figural representations

The philosophy of mathematics occupies an original position within the philosophy of science. On the one hand, the importance of mathematics in contemporary science is so great that, in principle, no philosophical research on science can do without the nature of mathematics and mathematical knowledge. For Bonnay and Dubucs (2011), at the horizon of the philosophy of mathematics, the philosophy of science rises fundamental questions, such as the possibility of completing the epistemology naturalization program, or the problem of the applicability of mathematics. On the other hand, the methodology of mathematics seems very far from the general methodology of science. In other words, the mathematician does not work in the laboratory; the classic problems of the general philosophy of science, which apply to empirical disciplines, referring, for example, to the question of confirmation, causality, or theoretical change, are not immediately transferable.

Bonnay and Dubucs (2011) state that when it comes to the epistemology of mathematics, it is necessary to explain of what the activity of mathematicians consists; in what sense it is a theoretical activity; what its objects are; what its methods are; and how everything fits into a global view of science, including natural sciences. Some philosophers consider that mathematics studies a domain of objects that exist regardless of us and that there are mathematical and physical objects, even if they are not the same type of objects. Others believe that none of this is true and claim that mathematical objects are just convenient fictions, or that we construct mathematical objects, or that mathematics only describes very abstract properties of experience. Some consider

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mathematical knowledge to be purely intellectual. Others think it is knowledge that is based on a form of intuition. Yet, others refuse to give it a special place and just want to talk about mathematical knowledge integrated into the whole science building.

Concerning the empiricist conception, Bonnay and Dubus (2011) assert that radical empiricism that bases mathematical truths on experience does not meet the problem of the rationalist, who must explain, to any mathematical axiom, what makes it a truth of accessible reason regardless of all experience. In reducing mathematical truths to empirical truths, radical empiricism does not explain the apparent modal and epistemic properties of the mathematical truths. Mathematical truths seem necessary and knowable regardless of experience, unlike contingent empirical truths. Furthermore, the distance between mathematical notions and experience makes the empirical deduction difficult.

Kant sought to recognize a role for intuition in mathematics, without this intuition making mathematical truths dependent on empirical content. In the famous texts of the Critique of Pure Reason and the Prolegomena, Kant (apud Bonnay and Dubucs, 2011) argues that mathematical propositions cannot be considered analytical propositions: there is more to the concept of four than the concept of the sum of two and two. For Kant, if we know that two and two forms four (on the decimal basis), it is because we go beyond the simple concept of adding two and two and resort to intuition, for example, counting with the fingers. Again, the whole problem is to understand how we can rely on an apparently empirical intuition to establish knowledge that is not empirical.

In Kant's terms, the problem is to understand the possibility of a priori synthetic judgments based on intuition and to assume the existence of a pure intuition of the forms of sensitivity, distinguishing two aspects of the phenomena: their form, which corresponds to how the phenomena are ordered in relation to others and their material, which corresponds to the sensation (BONNY; DUBUCS, 2011). The authors also point out that the forms of sensitivity, which are time and space, are given a priori; they do not depend on experience and are fundamental in the process of building experience. Arithmetic is based on pure intuition of time, while geometry is based on pure intuition of space. If the link between arithmetic and temporality only makes sense through the specificities of Kant's elaboration of the relations between consciousness and time, the connection between geometry and space is obviously less problematic, and the Kantian philosophy of geometry has some fidelity to the practice of geometricians.

Mathematical historians point out that Euclid's postulates indicate possibilities of construction: we can always draw a circle (empirically, using a compass), we can always

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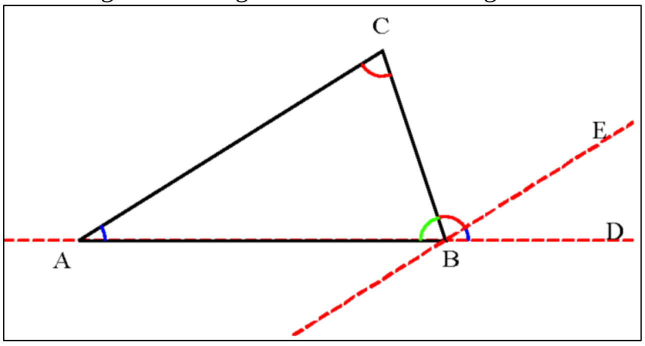
extend a line (empirically, using a ruler). Correlatively, Euclidean geometric demonstrations are based on auxiliary constructions.

For example, to demonstrate that the sum of the measures of the internal angles of a triangle is equal to the measure of a plane angle, we start from any triangle and draw a line parallel to one side, that passes through the vertex opposite that side. The demonstration is based on reasoning as of the initial figure, and the auxiliary constructions carried out. In this case, this reasoning will consist of using properties of the angles formed by the line drawn with the straight lines that support the other two sides of the triangle (in the order of the Euclid Elements demonstrations, these properties have already been demonstrated) (Figure 1).

Figure 1: Figure illustrating the reconfiguration of the initial figure to demonstrate the property



Source: Bonny and Dubucs (2011, p. 6)

The authors reinforce that mathematical intuition is at stake in these constructions, without which the demonstrations could not be carried out. However, the contingent characteristics of what is constructed are not and should not be used in the demonstration; otherwise, a necessary geometric proposal would not have been demonstrated. Kant believes the use of these constructions in proofs is legitimate because only the properties based on what can be done in space are maintained in the demonstration, rather than the empirical properties of the figures. Only the pure part of the empirical intuition is relevant in empirical intuition that underlies mathematical reasoning. The difficulties encountered by Kant's philosophy of mathematics "are proportional to its initial seductive strength. Those difficulties are due in part to the mysteries of the transcendental approach: what are the forms of sensitivity, why are they a priori, and what relationships do they have with the empirical constitution of the subject?" (BONNY; DUBUS, 2011, p. 69)

Based on Rouche (1989), we present several ways of demonstrating from elementary geometric properties to more abstract theorems and show that the mutations

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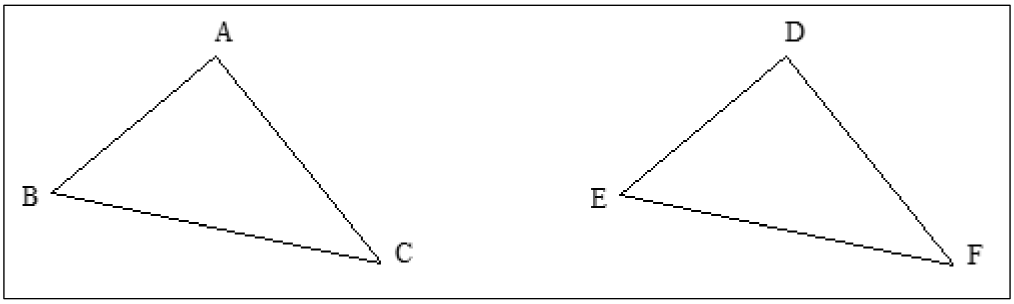
of the idea of proof go hand in hand with the transformation of the meaning of the forms of access to the meaning of the mathematical content treated.

We considered the first case of congruence of triangles as proposed by Euclides.

> First case of congruence of triangles: Two triangles are congruent if they have an equal angle between equal sides, each one to each one. Euclid's demonstration consists of considering two specific triangles (ABC and DEF) assuming $\hat{A} = \hat{B}$ and $AB = DE$ and $AC = DF$. Then, the first triangle is transported over the second so that A coincides with point D and that AB takes the direction of DE. We verify, then, successively, thanks to the hypotheses, that B coincides with E, that AC goes toward DF and, finally, that C coincides with F, which ends the demonstration (ROUCHE, 1989, p. 11, our translation)

Figure 2: comparison of two triangles



Source: Rouche (1989, p. 11)

Besides the two triangles of the figure, the theorem concerns all pairs of triangles that satisfy the assumptions. Therefore, if we find two triangles that we will know for whatever reason (and usually after a demonstration) that satisfy the hypotheses, we will see that they are congruent without having to overlap them. Thus, the theorem takes thought far beyond the limits of here and now, towards an infinity of triangles of all sizes and proportions.

For us, according to Rouche (1989) and Duval (2005), the two triangles play the role of representatives of the possible triangles. We are still a long way from mathematical thinking based on arbitrary symbols because representatives have a close relationship with the objects represented, which implies the possibility of easily passing from one to the other. The thought can engage without obstruction towards the object, imagining other cases in large numbers (ROUCHE, 1989). But it is, in fact, a possibility, not a spontaneous and constant process. It often happens that this possibility does not reach awareness and that, especially for beginners, attention remains entirely focused on the figure that accompanies the statement. As far as possible, in no way it gets in the way, the path of other likely figures is of little interest. No particular figure has a special value, and the possibility of always imagining other figures is important, without encountering obstacles. This possibility constitutes the theorem.

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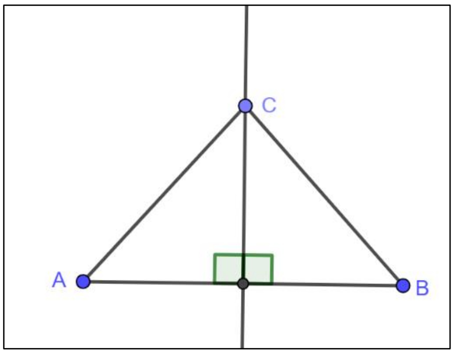
We analyzed, from another example taken from Rouche (1989), the circumstances that lead to reasoning, based on immediate thinking, given the following proposition.

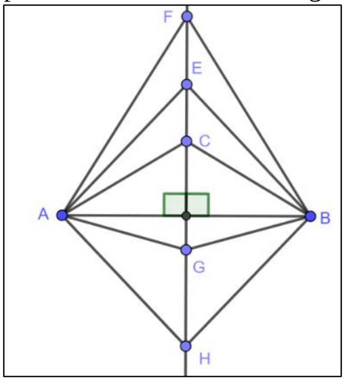
Proposition 2: The perpendicular bisector of a segment (that is, the line perpendicular to the segment in its middle) is the place of the points equidistant from its ends and vice versa.

Figure 3: A point in the perpendicular bisector of AB segment



Source: Adapted from Rouche (1989, p. 13)

Figure 4: Equidistance of the points of the perpendicular bisector of AB segment



Source: Adapted from Rouche (1989, p. 13)

The imagination embraces with a movement all the other figures that we could make from other points of the perpendicular bisector (Fig. 3) and all those that can be construct in the same way in other segments. Obviously, thought cannot go through all those figures, since they are infinite, but, when embarking on a journey, it effectively gets convinced that nothing would prevent it from going through new ones indefinitely. It potentially has access to every imaginable figure.

Rouche (1989) notes that a double condition seems necessary and sufficient for a proposition to be experienced as evident:

a) to discern at sight the implementation of a particular case;

b) thought is involved in the imagination without any problems in all possible cases.

The evidence subject to these two conditions is that which refers to the considered proposal that encompasses all cases that satisfy the hypotheses. When the attention does not go beyond the case considered, the feeling of obviousness is only due to condition a). In the context of geometry, cases are identified with figures.

Proposition 2 affirms the necessary concomitance of two properties: the one that defines the perpendicular bisector, and the equidistance property.

The author asserts that the inductions mentioned above do not refer to exact productions of the same figure, but, on the contrary, to all possible variants of one or

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another figure actually considered. Looking closely at Poincaré's view, induction does not lead to describing any quite new experience. It only expresses confidence in the possibility of reproducing exactly an experience, without changing anything. Our induction examples, on the contrary, provide information on a multitude of substantially different cases.

In this perspective, Globot (1907) asserts that geometric reasoning is never purely contemplative; it is active and constructive, and it is the constructive activity of the mind that produces a new result. A purely contemplative thought could not discover in its object anything but that same object. Moving from one property to another could result in the discovery of more unique and restricted propositions. General propositions, that are only truths when we just look at them, become rules when active and operational thinking that takes these truths as practical rules of our action takes place.

In the next topic, we provide reflections and criticisms about Poincaré's theory of mathematical demonstration.

3 The mathematical demonstration: Criticism of Poincaré's theory

To reflect on the concept of demonstration according to Poincaré, we rely on Globot (1907), who claims that

> logic is closely linked, on the one hand, to the theory of knowledge because we cannot discover the knowledge base without knowing exactly what must be constructed; on the other hand, to the psychology of the concept, to judgment and reasoning because, if it is possible to distinguish precisely the logical problem from the psychological problem, it is not possible to separate them (p. 265, our translation).

The author asserts that deductive reasoning is generally accepted as a syllogism, which seems to be considered by logicians as the only form of deductive reasoning. That mathematical sciences are deductive is also accepted. However, no mathematical demonstration is reduced to a compound syllogism. In this perspective, Globot (1907) states that:

> The chain of theorems leads to increasingly general propositions; algebra is more general than arithmetic, infinitesimal calculus is a generalization of elementary algebra, the geometry of the moderns is more general than the geometry of the ancients. Syllogism cannot be an instrument of generalization. Its fundamental rule, the Dictum de omni et nullo, prohibits this (p. 265, our translation).

He also states that the condition of validity of any syllogism is that the consequence must be contained in the principles. However, in mathematical demonstration, the consequence results from but is not contained in the principles. We

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cannot say that in an isosceles triangle, the congruence of the angles is contained in the congruence of the sides, nor the equality of the sides in the equality of the angles. The syllogism mechanism rests solely on the inclusion and exclusion relationships of the terms. Any mathematical demonstration establishes a necessary dependency relationship between heterogeneous properties (GLOBOT, 1907).

The syllogism is in any mathematical demonstration and has a well-defined function. It is used to apply a previously accepted principle or proposition to the case under consideration. “But it is never all reasoning. No demonstration consists of extracting a special proposition from a more general proposition that contains it” (GLOBOT, 1907, p. 266, our translation).

Globot (1907) also reinforces that the mathematician endeavors to arrive by the shortest possible route to the highest possible generalities, from those to even more general ones. They rarely retrace their steps; they do not enjoy making an inventory of all the partial truths contained in a more extensive truth unless they need to bring a remarkable property, which is generally called a corollary. It consists of formulating separately, because we will need it later, a property that, established during the demonstration or contained implicitly in the conclusion, does not need to be demonstrated separately. So, for the author,

> mathematical reasoning ranges from an admitted property to a heterogeneous property (in the isosceles triangle, from equality of sides to equality of angles) or from a special property to a general property (from the sum of the measures of the internal angles of a triangle to the sum of the angles of the polygon), never from general property to special property (GLOBOT, 1907, p. 266).

The author states that there is an advantage in drawing pure, more general, more abstract relations from the spatial intuitions in which we first consider them, and in deriving them from each other in an independent way to look at the figures.

Poincaré finds the reasoning by recurrence in the demonstration of the addition and multiplication rules, that is, the most elementary rules of arithmetic or algebraic calculation. Poincaré (apud GLOBOT, 1907) asserts that

> This calculation is an instrument of transformation that assists in many diverse combinations than simple syllogism; but it is still a purely analytical instrument and unable to teach us anything new. If mathematics had no others, it would be immediately blocked in its development; but it resorts to the same process again, that is, to reason by recurrence and can continue marching forward. - At each step, if we look closely, we find this way of reasoning in the simple form we have just given it, or in a more or less modified way. So this is the mathematical reasoning par excellence...” (p. 268, our translation).

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We do not find reasoning by recurrence in all mathematical demonstrations. The author claims that Poincaré seems to implicitly recognize that this type of reasoning only accidentally intervenes in algebra, but algebra is purely “analytical,” and only operates “transformations.

Reasoning by recurrence plays an essential role in algebra, which allowed it to expand its domain; it is present at the beginning of the infinitesimal analysis. It intervenes whenever mathematics crosses a trench and attaches new territory (GLOBOT, 1907).

> While it is limited to exploring the domain conquered, without expanding it, it does not make use of it, but neither does it advance, it transforms. I believe that Poincaré is mistaken. Reasoning by recurrence is a very special and recognizable way; there are real and general algebra demonstration that cannot be reduced to this. Algebraic transformations can be used to demonstrate new propositions; they do not consist of stepping on the spot; they advance (p. 269).

Globot (19007) advances two reasons over Poincaré’s thesis that recurrence is not the only mode of general and generalizing demonstration. The first reason is that this type of reasoning applies only to the series of integers. Mathematics has become increasingly arithmeticized. Poincaré builds continuity, the dimensions of space, reduces geometry to the calculation of functions, and states that: “the geometer makes geometry with extension, as is done with chalk; therefore, we must be careful to give much importance to incidents that generally have nothing more than the whiteness of the chalk” (POINCARÉ, 1906, apud GLOBOT, 1907, p. 269, our translation).

The second reason is even more decisive, says Globot, because the reasoning by recurrence contains a demonstration that this process hardly takes into account. The author states that the geometric demonstration is generalized in two ways: first, every demonstration advances from the singular to the general and consists of establishing a necessary relationship between two heterogeneous properties, which cannot be done by any syllogism or by any composition of syllogisms. Second, some demonstrations go from the special toward the general, which also cannot be explained by syllogistic reasoning. For example, to demonstrate that in an isosceles triangle the angles opposite to the congruent sides have the same measure, we exfoliate the triangle, detach it from itself by thought, and reapply it, turning it upside down, in the line that we suppose to have left on the board.

In this perspective, the Gestalt laws made it possible to theorize, scientifically, the visual reading system that allows us to analyze and interpret objects, considering that some forms can facilitate or hinder their perception, depending on the factors of composition in this way. Duval (2012a) shows four different ways of apprehending a

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figure: the perceptive (immediate visual recognition of the shape), the operative (reconfiguration operation), the discursive (indications contained in the statement) and the sequential (indications to construct a figure). They are independent of each other, but the resolution of a problem requires the transition from one type to another type of apprehension.

Duval (2005) asserts that to see a figure, in geometry, it is necessary to dissociate what refers to size and, therefore, what depends on the scale of magnitude in which the act of seeing occurs, and what refers to the discriminated forms, that are independent of the scale. The relationship with the figures, that is, the way of seeing what they show, concerns the discrimination of shapes and not the size or changes in the scale of size. It is Poincaré's (1963) perspective as regards "geometric intuition:"

> When in a metric geometry theorem, we appeal to this intuition, it is because it is impossible to study the metric properties of a figure without considering its qualitative properties, i.e., say those that are the appropriate object of Situs Analysis.... it is to promote this intuition that the geometer needs to draw the figures, or at least represent them mentally. However, if these metric or projective properties of those figures are used cheaply, focusing only on their purely qualitative properties, it is because this is where geometric intuition really intervenes (p. 134-135, apud DUVAL, 2005, p. 6-7, our translation).

The discrimination of those "purely qualitative properties" is one of the most critical aspects of the process of demonstration in geometry. One of the problems is the perception that, according to Duval (2005), it works without any dissociation between magnitude and visual discrimination of forms, and, above all, it imposes a common way of seeing that goes against the two ways of seeing figures that are requested by geometric demonstrations: one centered on the possibility or not of constructing the figures with the help of instruments; and the other, centered on their heuristic enrichment to reveal shapes that are not those that the eye sees in them. We agree with the author when he asserts that it is difficult to transpose the transition from the usual functioning of the perception of forms to those two ways of seeing, especially the second. However, they are only the superficial manifestation of a third way, which constitutes the cognitive mechanism of mathematical visualization: the dimensional deconstruction of forms (DUVAL, 2005). In this perspective, the author reinforces that:

> The construction of figures, or their heuristic use, only makes sense to the extent that they are part of this functioning of mathematical visualization because, with this third way of seeing, space is no longer approached from the perspective of magnitudes and changing scales of magnitudes, nor under the discriminatory aspect of topological and similar properties of forms. The way to see a figure is done according to its dimensions and the change in the number

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> of its dimensions. The altered number of dimensions is at the center of the geometric appearance in the figures (p. 7, our translation).

Chart 1 shows four ways Duval (2005) presents of looking at a figure.

Chart 1: Four classic entries in geometry

	BOTANIST	**SURVEYOR**	**BUILDER**	**INVENTOR**
1 Type of operation on *VISUAL FORMS*, required by the activity proposed.	Recognizing shapes from the visual qualities of an outline*: A particular shape is privileged as TYPICAL.*	Measuring the sides of a surface: *on an AREA or on a DRAWING (magnitude scale variation and then measurement procedure).*	*Decomposing a shape into constructable tracings with the help of an instrument* (often) needs passing through *AUXILIARY TRACINGS* that do not belong to the "final" figure.	*Transform some shapes into others.* To begin these transformations, it is necessary to add the final figure *REORGANIZING TRACES.*
2. How *GEOMETRIC PROPERTIES* are mobilized *in relation to this type of operation.*	*No links between different properties (no mathematical definition possible)*	Properties *are the choice criteria* to take measurements. They are only useful if they lead to *a formula that allows a calculation*	As *setbacks for a construction order.* Some properties are obtained by *a single tracing operation*, while others require several operations	Implicitly by reference to a more complex network (a network of lines for plane geometry or a network of plane intersections...) than the initial figure.

Source: Duval (2005, p. 9, our translation)

Through analyzing the four ways of looking at a figure, we infer that the botanist gaze allows us to recognize the contour of shapes. It is the most immediate and evident entry; it refers to a "qualitative look." The author states that it concerns "[...] observing differences between two shapes that have certain similarities (a square and a rectangle) and noticing some similarities between different shapes (a square and a parallelogram)" (DUVAL, 2005, p. 10). That is, the properties that stand out in the figures are the visual contour characteristics. Bearing in mind that the action of recognizing shapes can be done in another way, it is not configured as a geometric activity (DUVAL, 2005).

The surveyor's gaze, as we will see in this study, played an essential role in the historical trajectory of geometry, as we find through the many examples in Greek geometry. In this way of looking, the activity of making measurements on a terrain surface and passing them to the paper plane is emphasized. Geometric properties are mobilized around measures that are only useful if they lead to some calculation procedure. Therefore, it is a matter of correlating two scales of magnitudes. "Now, the fact of putting in correspondence is nothing natural or evident, as there is no common procedure to

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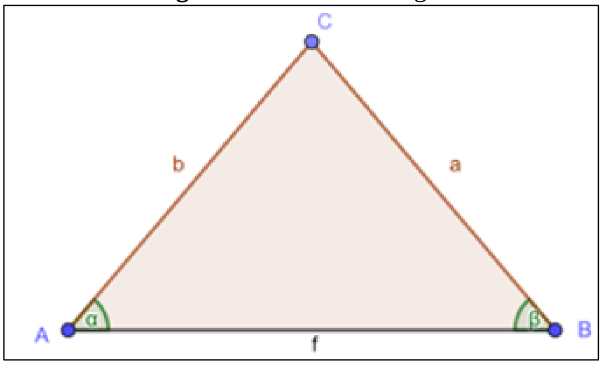
measure the real distances on the terrain and to measure the widths of the traces of a drawing" (DUVAL, 2005, p. 10).

The dimensional deconstruction is the central axis of the geometry visualization process. The dimensional awareness of shapes and their discursive operations allow visualization and discourse to be in synergy (DUVAL, 2011). The passage between visualization and discourse is closely connected to a dimensional change of shapes to enable us to recognize the geometric objects in each of the two registers, remembering that each semiotic representation for the same geometric object can address different contents (DUVAL, 2005).

Figure 5: Isosceles triangle



Source: author's construction using Geogebra

From the dimensional deconstruction of figure 5, we observe that the angle ($\widehat{ACB}$), between the congruent sides ($\overline{AC}$ *e* $\overline{CB}$), necessarily coincides with its own trace, that each side of this angle coincides with the trace on the other side that is congruent to it. The coincidence of the third side results from the principle that two points can only be joined by a single straight line; it is the only syllogism involved in the demonstration. Finally, we note that each of the angles opposite the congruent sides coincides with the trace of the other. As we can see, the demonstration consisted of an operation and observation of the result obtained (GLOBOT, 1907).

Actually, this process, according to Duval (1995, apud ALMOULOUD *et al.*, 2004), involves three types of cognitive processes, which perform specific epistemological functions:

- the visualization process for heuristic exploration of a complex situation;
- the construction of configurations, which can be worked as a model in which the actions performed represented and the results observed are linked to the mathematical objects represented;
- reasoning, which is the process that leads to proof and explanation.

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Duval (1995, apud ALMOULOUD *et al*., 2004, p. 126) states that "those processes are intertwined, and their synergy is cognitively necessary for proficiency in Geometry." However, they can be carried out independently. For example, the construction can lead to visualization, but it does not depend on the construction. Visualization, on the other hand, can contribute to reasoning, as well as lead to making mistakes (DUVAL, 1995).

To arrive at the expected result in the example in figure 5, we articulated the perceptual, operative, and discursive apprehensions. Perceptual apprehension allows us to identify or recognize, at first glance, a shape or an object, whether in a 2D or 3D. Duval (1994, p. 124) asserts that "This is done by cognitive processing carried out automatically, therefore unconsciously. That is why the shape of a figure, or those that compose it, is recognized from the first time, and that recognition remains stable." Depending on their dimension, those elements can be traces (following the law of closure and continuity), points (discrete or continuous), or zones (characterized by their contour).

The operative apprehension of figures is the perception of the organization and reorganization of the set of forms of a figure that lead to the performance of various reconfiguration operations through physical or mental manipulations on the whole or part of the figure. It is centered on the possible modifications of an initial figure and the possible reorganizations of these modifications. For each type of modification, there are several possible operations" (DUVAL, 2012b, p. 125).

The discursive apprehension of a figure "[...] is equivalent to immerse, according to the indications of a statement, a particular geometric figure in a semantic network, which is, at the same time, more complex and more stable" (DUVAL, 2012b, p. 135). This, because the figure alone cannot represent all its characteristics, it needs a verbal indication to anchor the figure as a representation of the mathematical object. However, a figure's privilege is perceived over language. We recognize that the representation of an image is obvious and it is possible to make all statements about it understandable, that is, "we postulate that the articulation between 'image' and 'language' would occur spontaneously" (DUVAL, 2003, p. 39), reinforcing the idea that the figure alone is capable of leading to the interpretation of a situation.

What was done in the example in figure 5 was not a manual but a mental operation, and it is not a physical - as could be done with measuring instruments- but a logical observation. In intuitive geometry, all geometric demonstrations are done based on examples. It is just that we demonstrate by operating; however, an operation

ISSN 2525-8222
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(construction, superposition, rotation, etc.) can only be executed, even mentally, and the result of an operation can be observed in the same way, only in a singular figure (GLOBOT, 1907).

The author reinforces that this result, although observed, is necessary, and the operation was carried out according to the rules. The rules of the operation are, initially, the general definitions and the special hypotheses that determine the question, i.e., the conventions that the mind has made with itself and with which it is connected, and whenever there are reasons to resort to the propositions previously established. The result observed is necessary to the extent that the application of the rules determines it.

It remains contingent and modifiable, as it depends on the singularities of the example chosen, and that is why it is general. When reasoning about a figure, the researcher always keeps in mind the distinction between the properties of that figure formally stated in the hypothesis and those that, if not specified, remain indefinitely variable. The operation,

> which is regulated only by the first, can be repeated, with the same result, in any different figure that performs the hypothesis, whatever its unique properties. The operation that consists of detaching the isosceles triangle from the table plane and reapplying it in its own trace can always be repeated and will always give the same result, the coincidence of the opposite angles to the equal sides, in any triangle, because I took as a rule for this operation the equality of two sides, not the absolute value, nor, except for this assumption, the relative value of the sides or the angles (GLOBOT, 1907, p. 372, our translation).

The same author asserts that it may seem surprising that a finding is inherent because when it is an empirical observation, the observing scientist, the physicist, for example, records the manifestations of forces that are foreign to them. Nature operates before their eyes, according to rules or laws which they ignore and which are precisely the object of “research.” The surveyor, on the contrary, operates according to the rules he/she knows, of which they constantly feels the restriction that always guides him/her and often resists him/her; and, in fact, he/she never has any other guarantee of the need for his/her results other than the awareness of having observed them.

Based on Globot (1907), we affirm that, in mathematical reasoning, generality is a consequence of necessity, an essential character of deductive reasoning, which consists, first, of realizing that a relationship is necessary, therefore, general. Inductive reasoning, on the contrary, consists of establishing a series of operations, at the end of which we observe facts and empirical findings from which we infer that a relationship is constant and necessary, since chance and contingency cannot produce perfect uniformity. But this

ISSN 2525-8222
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need “is only inferred; it remains hidden, illusory, not perceived by the mind, as long as it sticks to inductive reasoning” (GLOBOT, 1907, p. 172).

The author stresses that it is false to believe that all deductive reasoning proceeds “from the general to the particular;”

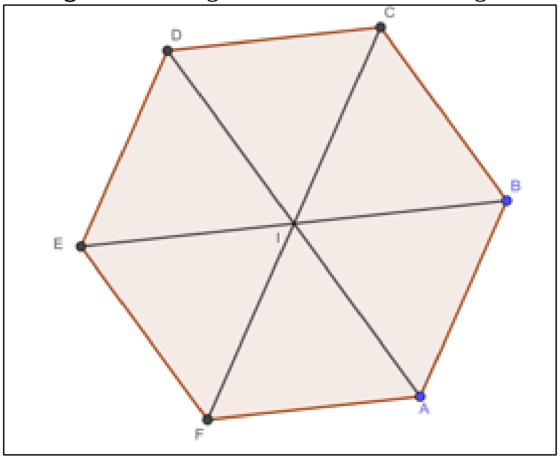
> it is false that any reasoning that proceeds “from the particular to the general” is inductive. If we call induction any reasoning that generalizes, there will be nothing left that we can call deduction, because any real reasoning makes us acquire new knowledge. Syllogism is not, strictly speaking, a reasoning, but a part of reasoning, a time, an articulation of reasoning. When, appealing to a general principle, we feel the need to make it, so to speak, the currency, retain only a part of it, what concerns the object on which we are focused, is that we want to go further and, using the principle, bring a new consequence, increase our knowledge. The reasoning is not complete until the mind, taking the principle as specialized as the rule of its operation, has built a new property (GLOBOT, 1907, p. 273, our translation).

We agree with the author when he states that the geometric demonstration also generalizes in another way, going from the special to the general. It often happens that a proposition cannot be demonstrated immediately in its general form; we first demonstrate this for a special privileged case, for which we have reduced the general case. The special privileged case has this advantage, because of its special properties, it allows constructions or operations that cannot be performed in the general case. So, first, we demonstrate that the sum of the measures of the internal angles of a triangle is equal to 180º because a given construction (Figure 1) that assimilates the angles of the triangle to adjacent angles on one side of a line is possible.

The property demonstrated for the special privileged case cannot be immediately extended to the general case, since the demonstration process took some special property of the privileged case as a rule. A demonstration is necessary, but the theory explained above clarifies this perfectly, because this demonstration always consists in constructing the general case from the special case. Thus, in the example just mentioned, the sum of the measures of the angles of a polygon is constructed with the sum of the measures of all the angles of all the triangles into which it was decomposed (Figure 6).

ISSN 2525-8222
DOI: http://dx.doi.org/10.33361/RPQ.2020.v.8.n.18.344

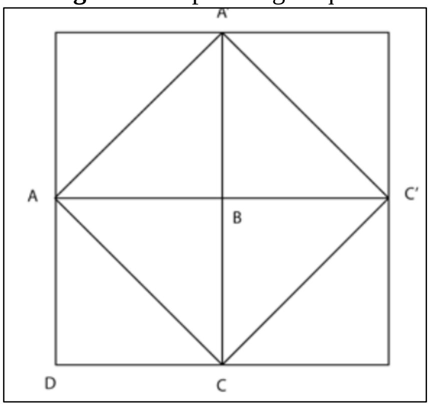
Figure 6: Hexagon divided into six triangles



Source: Own construction using Geogebra

Another famous example, taken from Chauve (2006), is the duplication of the square, whose demonstration was made by Pythagoras, and which Plato staged in Menon. To demonstrate that the square which has as its side the diagonal AC of another ABCD square will have double the surface of the latter, we constructed the figure. Then, we counted the congruent triangles (Figure 7) to verify that there are two in the square provided (ABC and ACD) and four in the square constructed (ABC, ABA´, A´BC´ and C´BC).

Figure 7: Duplicating a square



Source: Chauve (2006, p. 3)

Chauve (2006) states that in this rudimentary form, the demonstration consists of assembling something in a figure. However, it is not a figure that would be found among the things that can be observed. The figure we are showing is a figure that we have built, and that does not appear only in this construction. And showing something in this figure is not just looking at it, no matter how carefully, to try to perceive it, but reasoning with attention, in particular, to establish equality between its elements.

ISSN 2525-8222
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The author asserts that, at the beginning of geometry, the demonstration consists of showing not by observing and looking, but by constructing a figure to be able to reason about the construction. The geometric object is not what we make appear visually, but a figure that we represent in the mind and that is like the background of the figure that we draw (CHAUVE, 2006; DUVAL, 2005). Platon (République VI, 510 b-e, apud Chauve, 2006) said that this object is "supposed" and that, for example, under the square we draw, there is the "square itself." When appearing in the field of geometry, the notion of demonstration requires that we distinguish and separate mathematical things that we design and form, through thought, concrete and "visible" things that we can observe around us. From the beginning, the notion of demonstration is supportive of a philosophical conception of the nature of mathematical things, so to touch that notion is to modify that conception. And that is just what will happen.

The examples presented are all from Euclidean geometry. However,

> Poincaré will tell me, without a doubt, that his theory correctly highlights the imperfection of this intuitive geometry, which painfully reaches general propositions and only relatively general ones. He made a constant appeal to intuition and never freed himself from considering singular figures. It needs a blackboard and chalk, not like algebra, to write its reasoning, but to make the very objects on which it reasons. It follows that it can proceed only through observations, then generalize as it can. Thus, like analysis, it never succeeds in establishing purely abstract relationships, independent of all intuition, empirical or not, and which remain true in themselves, even when we have no intuition to which we can apply them (GLOBOT, 1907, p. 274, our translation).

Poincaré would say, according to Globot (1907), that the procedure above would result in an observation procedure; then, eventually, generalizations. Thus, like analysis, it never succeeds in establishing purely abstract relationships, independent of all intuition, empirical or not, and which remain true in themselves, even when we have no intuition to which we can apply them. The true mathematician would replace geometry by calculating functions of three independent variables and since nothing compels him/her - if not for the sake of convenience- to limit himself/herself to that number of three independent variables, he/she would conceive Euclidean geometry as a special case of "a general geometry."

Thanks to algebraic analysis, she extracts the abstract relations from the spatial intuitions in which they were first considered and deduces them directly from each other. Undoubtedly, intuitive geometry retains all its pedagogical value, since beginners must redo "quickly, but without skipping steps, the path taken slowly by the founders of science" (GLOBOT, 1907, 275).

ISSN 2525-8222
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The author further asserts (about what Poincaré would say) that the trained mathematician is no longer satisfied with these mental operations, which, without needing to be performed manually, are, however, the closest to manual operations and logical observations that, without being empirical observations, cannot do without intuition. Analytical methods are both more abstract, more general, and more rigorous. Intuitive geometry is just a mathematical science. However, all branches of pure mathematics are reduced to the calculation of integers, and the rules for calculating the integers, like the rules for each of their successive extensions, are demonstrated by the recurrence method. This is, therefore, the only real and purely mathematical reasoning.

For Globot (1907, 275, our translation)

> the theory above applies exactly to all arithmetic, algebra and infinitesimal analysis demonstrations, while reasoning by induction, unless a disguise hides it, it is a special and relatively rare mode of demonstration; I also reply that it applies to reasoning by the induction itself and that it is necessary to explain it.

He asserts that we are mistaken when we say that algebraic calculus, consisting of simple transformations, is purely analytical and does not introduce anything new. Algebra has no other object than the form of algebraic expressions, and its demonstrations cannot relate to anything else. All algebra propositions state that a given form necessarily results in another form. To establish those necessary relations between heterogeneous forms, we did not discover that the second was contained in the first (if that were the case, there would be no demonstration to be done, but a simple application, in other words, no reasoning, but a simple syllogism). The demonstration consists of creating the shape from the first. Therefore, the algebraic calculus is precisely comparable to the geometric construction. The author reinforces that

> We are very inclined to see transformations in formulas and geometric constructions as incidental operations that prepare and precede reasoning or that follow and result from it; they are its constituent and essential elements. The constructive operation brings a new result, the rule guarantees that it is necessary. There is no proposition of arithmetic or algebra that is not demonstrated through an operation or a series of operations. Any proposition to be demonstrated consists of a hypothesis and a consequence; the consequence is not identical to the hypothesis, nor is it contained in the hypothesis; it is, therefore, heterogeneous; the only way to demonstrate what it does is to construct the consequence from the hypothesis (GLOBOT, 1907, p. 275-276, our translation).

It states that the knowledge of the result of an arithmetic or algebraic operation is a *logical finding*; after getting the successive portions of a sum, of a difference, of a product, of a quotient one by one, we see the result "found" by a "reading." We believe that the result obtained is necessary because we are convinced that we operate according

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to the rules, which are: 1st, the logical conventions, that is, definitions and hypotheses; 2nd, the general propositions previously demonstrated. But the result is necessary only to the extent that it was determined by the rules. About the unique properties, not specified by the premises, quantities, or forms in which we operate, the result remains entirely flexible and indefinitely changeable. Like the surveyor, the analyst discerns effortlessly and painlessly in his/her formulas what is necessary and what is not. In this perspective, Globot (1907) reinforces his argument based on the following example:

> To demonstrate that the square of the sum of two quantities is equal to the sum of the squares of each of them, plus the double of their product, we performed the operation $(a + b)^2$; we observed the form of the result; and, as we took as a rule, doing the operation, the form of the expression a + b, and in no way the value or the nature of the two quantities a and b, we know that, whatever these two quantities are, numerical or not, known or unknown, commensurable or immeasurable, in any way they are composed, we will always have a result in the same form (p.276).

He concluded that the arithmetic, algebraic, and analytical demonstration is, therefore, of the same nature as the geometric demonstration. Among all possible hypotheses, the mathematician chooses those that will lead him/her to practically usable results. Mathematics is not the knowledge of any part of nature. Still, it aims to provide the natural sciences with a flexible and rigorous language, suitable for expressing the relationships between them and their dependence on each other. It turns out that several equally possible hypotheses can also lead to the expression of natural laws; the mathematician then chooses those that lead us through the shortest path to the simplest and most manageable expressions. From there, the value of science: among several hypotheses, among countless hypotheses of equal logical value, we give preference to this, not because it is true but because it is the most convenient one.

4 The philosophy of demonstration in geometry

The reflections made in this part are mainly supported by Chauve (2006). The author asserts that the demonstrative approach, to which we owe the first geometric laws, remains unsatisfactory, because, although we understand that it is necessary to appeal to constructions and not to observations, the need for the construction that must be made to demonstrate does not appear. In this perspective, the author asserts that:

> With Euclid, geometry clarified the laws that the constructions that serve to demonstrate a geometric law obey. We move from the geometric laws we demonstrate to the laws of geometry with which we demonstrate. We explain the laws of geometric thinking, the laws of the demonstrative approach in geometry. This development of geometry has actually three aspects: it focuses on the fundamental principles and notions of geometry; clarifies the

ISSN 2525-8222
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> mathematical meaning of geometry; brings up a philosophical presumption (CHAUVE, 2006, p. 6).

He claims that the Euclidean elaboration of geometry leads to clarifying and codifying the very geometric thought, i.e., what makes it geometry. It makes intelligible not only the objects of geometric knowledge but also the knowledge of those objects. Until then, geometry studied the properties of geometric objects, spontaneously calling the consideration of points, lines, planes, etc., without asking about what this point, this line, this plane, this angle, etc. is. Euclides actualizes, clarifies and corrects these concepts and the fundamental rules that govern their use in constructions. Thus, the mathematical meaning of geometry becomes clearer, since we move from considering the figures where we saw the image of the geometric things, to considering magnitudes that can be constructed with a ruler and compass in space. The construction possibilities are expressed by postulates and the rules of reasoning about magnitudes, by axioms. Chauve (2006) argues that:

> These possibilities and rules that make up geometric thought come to the fore, so that it is the method that determines the object, that delimits the field of geometry and limits its demonstrative possibilities. Euclid's geometry is impotent in the face of insoluble problems due to constructions using a ruler and compass: this is the case of the problem of Delos (god Apollo's birthplace, an island where an altar twice as large was required to appease the gods. It is the problem of cube duplication), problem of the trisection of the angle and the problem of quadratures (Archimedes will dominate it by the method of exhaustion which is no longer the "application of areas" that characterizes Euclid's geometry) (p.6, our translation).

For Kant[2] (1787, apud CHAUVE, 2006), geometry proceeds by "construction of conceptions," that is, to demonstrate, it is necessary to proceed with constructions. To the definition of a geometric object, we must associate a construction, and it is from the dimensional configurations of the figure that we must reason, because if we reason based solely on the definition, to try to deduce something, we will achieve nothing. For example, we cannot deduce, from the triangle definition, the sum of the measures of its angles, without reasoning about the construction of the figure (example of figure 1). To demonstrate, we need to construct a figure; but to construct a figure, we must have a representation of the mathematical object, which is an a priori representation, which is not given to us by what we observe or what we find. It cannot be the empirical representation of the contour of a figure. It can only be the representation we have of him in space, that is, in "pure intuition," as Kant says, in which the concrete things that we

[2] Emmaneul Kant (1787) . Critique de la Raison pure, introduction, trad. Jules Barni, revue par Archambault, éd. G. F. Flammarion.

ISSN 2525-8222
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can see in the space around us are disregarded, but where we represent the extension itself. This representation is the final condition of geometry.

Kant (1787, apud CHAUVE, 2006) argues that axioms and postulates such that “the whole is greater than the part,” “between two points, passes only one line” etc., fall under a principle called by Kant the “axiom of intuition” and states that “all intuitions are extensive magnitudes” (one can only imagine something in the form of magnitude in space).

> This principle, although qualified as “mathematical,” is not itself an axiom of mathematics, it is, Kant specifies, “the foundation of the possibility of axioms” by virtue of which “these axioms themselves are not [...] admitted to the topic only because they can be represented in pure intuition. The properties of geometric objects are accessible only in pure intuition, that of space, the foundation of the laws that govern geometric construction procedures. This pure intuition controls acts of representation and is constitutive of the possibility and power that our mind has to have representations, therefore, geometry is based on a pure intuition that forces the researcher to submit to the fundamental requirements of representation. A condition is imposed on the surveyor’s speech: it is necessary that it is not only deductive but that it submits to the possibilities of representation. The demonstration requires representation (CHAUVE, 2006, p. 7, our translation).

The author states that Euclides clearly identified the geometric thought underlying the demonstration by constructing figures, but he did not take that thought as an object of geometry. On the contrary, this is what Gauss, Bolyai, and Lobachevski (1935) will do. It is no longer a matter of demonstrating something using a postulate that commands a geometric construction, but of trying to demonstrate the very postulate that serves to demonstrate. The demonstration means are taken as a demonstration object, so that, in the end, it is the geometric demonstration that will eventually become the object of geometry. A new concept of geometry will be born, and we are witnessing the disconcerting appearance of a new geometry. Gauss calls it “non-Euclidean geometry.” It sets off with the idea that, starting from a point outside a line, there are several “non-secant” lines that do not cross that line. Lobatchevski (1935), for example, resuming the construction corresponding to the formulation of the fifth postulate, founded his thought in the following way, based on figure 8:

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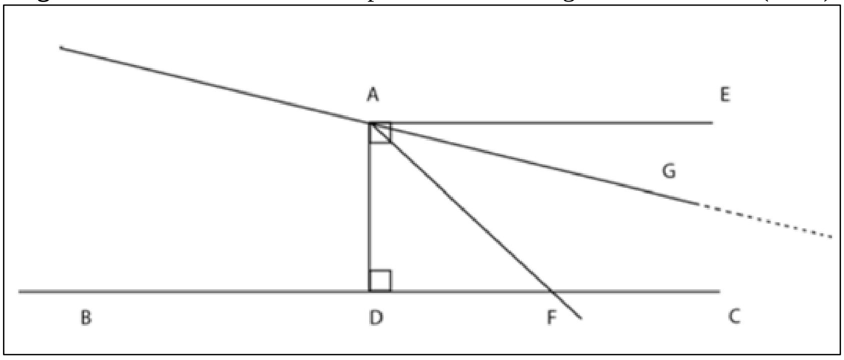
Figure 8: Illustration of the fifth postulate according to Lobatchevski (1935)



Source: Chauve (2006, p. 8)

Chauve (2006) comments on Lobatchevski's ideas as follows:

> From a point A outside a line BC, we lowered the line AD perpendicular to that line. From A, we drew a line AE perpendicular to AD. At the right angle EAD, all lines that start at A toward C and E cut the DC line, such as AF; or others, such as AE, do not cut the DC line. And Lobatchevski adds: "In the uncertainty, if the perpendicular EA is the only straight line that does not cut the straight CD, we admit the possibility that there are still other lines, such as AG, that do not cross the straight CD, no matter how far they are extended." What is the reason for this uncertainty that suggests that there may be other lines that do not cross the CD line? This is because, as we have pointed out, the fifth postulate does not formulate, like the others, a rule that expresses the possibility of carrying out a construction; rather, it formulates a rule that suddenly involves a statement about the result of a construction (the perpendicular AE will be the only one that will not cut DC) (p. 8, our translation).

The development of this geometric idea about the possibility of building other lines gave rise, when it appeared, to a mathematical and philosophical interpretation that we can no longer admit. Chauve (2006) asserts that, from a mathematical point of view, Gauss saw in this geometry a "non-Euclidean geometry."

> The term "non-Euclidean geometry" means that the refusal of Euclid's postulate does not contradict the other Euclidean axioms and postulates, so that if non-Euclidean geometry were contradictory, so would Euclidean geometry. When we speak of non-Euclidean geometry, we simply mean that the Euclidean postulate of parallels is independent of the other postulates and axioms, and nothing more. So, from a philosophical point of view, we no longer ask ourselves whether the postulate about the uniqueness of the parallel is true or false: we no longer interfere with the truth or the imaginary character of the new geometry. This geometry forces us to abandon the very idea that geometry is a representation of the real or the imaginary space; that representation was pure intuition (CHAUVE, 2006, p. 9, our translation).

It also ensures that in geometry, there is no representation of space, there is a structure of space. Geometry is actually an axiomatic in which we ignore all representation, and in which the word "space" designates a structure, i.e., a system of axioms and deductions. Axiomatizing a theory consists of rigorously formulating primary propositions that allow the theorems to be deduced without having to consider the nature

ISSN 2525-8222
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of the theory's objects. Those propositions are called first "axioms," but that word no longer has the restricted meaning Euclid gave it. Chauve (2006, p. 9) calls

> "demonstration" the pure deduction of the theorem of the axioms. It means that in those demonstrations, we disregard representations of geometric objects that seem to evoke the discourse of geometry or that would give us constructions in space. Those representations would be pure and a priori. Let us stick strictly to what we can deduce from the axioms according to the rules of the only logical-mathematical syntax of the geometric discourse. We must know that, when the geometer speaks of space, he/she speaks of a pure mathematical concept, that is, of a system of axioms and logical relations of the deductibility of propositions of those axioms.

Chauve also notes that the geometer does not speak of pure intuition, that would be an a priori form of objective representation. He no longer sees the possibilities of pure constructions that authorize this intuitive form, but he considers the possibilities of pure deductions of propositions due to the logical form and syntax of the geometric discourse. In axiomatized geometry, deduction replaces construction. For what is constructible, we substitute what is deductible in the syntax of the geometric discourse.

To reinforce this, we rely on Chauve's (2006, p. 10) example, that replaces straight and parallel lines with the syntax of the expressions with which we speak of parallelism. He considers an E set with four elements E = {a, b, c, d} and the pairs (a, b), (a, c), (, d), (b, c), (b, d), (c, d). Chauve denotes the pairs by D, D´, etc. He use the word "plane" to designate the set E, the word "point" to designate its elements and the "line" or "straight line" for a pair of points. He formulated two axioms to give this set a structure:

Axiom 1 (called incidence axiom): if the intersection of D and D´ contains at least two elements, then D = D'. Which means that through two "distinct" points "one and only one" straight line passes. When the intersection is a point, we can use the expression "secant lines."

Axiom 2 (axiom of parallelism): whatever D, if x (x being an element of E) does not belong to D, then there is one and only one D´, such that x belongs to D´ and the intersection of D and D´ is empty. That is, by a "point" x "outside" of a "line" D, there is one and only one D´ line parallel to D.

> In other words, in the case of the "4-point plane," we can use "straight lines" and "parallelism" in the sense of Euclid. It is certainly a strange "plane" that has no extension, strange "points" that are nowhere in space, strange "straight lines" that are not straight and that have no distance or length. But the strangest thing is that what we said about those strange things is exactly what is said about the plane, points, lines when we talk about parallelism. And what we say is not what we imagine when we speak. Therefore, we have the syntax of expressions with which we can speak of points, lines and parallels, but without having to imagine a geometric construction in space with straight lines that we draw, that intersect or extend to infinity without being interpreted. Not only do we not need to imagine such a construction, but we cannot imagine it either: in

ISSN 2525-8222
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> the “4-point plane,” the “plane “has no longer much to do with the representation of a flat surface, any more than “straight lines” with straight traces (CHAUVE, 2006, p. 10, our translation).

However, the axioms that were given by the author constitute well in geometric discourse the syntax that makes it possible to speak of parallelism and to which Euclidean reasoning involving parallelism can refer. “The geometric thought of parallelism” is not in what I represent when I speak of parallel lines, but in what I say when I speak of it. It is in the syntax of the logical-conjunction of what I say. There is no question that the terms “straight line” and “point” evoke geometric representations that lead to the non-recognition of this syntax of the geometric discourse.

> It just means that the representation of the line we have in mind - a straight line - carries other structures: order, measure (length), continuity. But, on the one hand, they still are and will always be structures to which this representation is reduced and, on the other hand, we do not need those structures to reason about the parallelism of the lines. To be convinced, it will be necessary to correct, rectify, purify, specify, reorganize the Euclidean discourse, classify the “axioms,” delimit precisely the field of the propositions to which they refer and determine exactly how they intervene in the manifestations (CHAUVE, 2006, p. 11, our translation).

The author asserts that this work was done by David Hilbert (1899), in *The Foundations of Geometry*, where he presented the first rigorous axiomatization of Euclid’s geometry. It is he who, to make his students understand that in an axiomatized geometry, structure replaces representation, said as if joking, that “we should be able to speak of geometry, tables, chairs, and beer mugs, instead of points, straight lines, and planes.”

Hilbert (1899 apud CHAUVE, 1990, p. 11) explains the “general requirements and conditions that must be satisfied by solving a mathematical problem.” He was trying to explain what mathematical reasoning should be, that is, the demonstration. In a demonstration, “the solution [...] must be obtained by a finite number of conclusions and must be based on a finite number of hypotheses provided by the very problem and grounded in each case with precision” (CHAUVE, 1990, p. 11). Thus, the demonstration, in its rigor, requires a “logical deduction by means of a finite number of conclusions.” This idea of demonstrative rigor meets an intellectual requirement that goes beyond the domain of mathematics. For Hilbert (1899), it represents a philosophical idea: the rigor in the demonstration “corresponds, he says, to a general philosophical need of our understanding.”

Chauve (2006) states that assuming a finite number and making a finite number of deductions, means, when facing a problem, clarifying and formulating exactly, in

ISSN 2525-8222
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purely logical-mathematical terms, the propositions that make it possible to solve the problem, that is, that allow us to proceed in a purely deductive manner, leading to the conclusion in a finite number of deductions. In geometry, for example, it will be necessary that the figures and constructions that we used to make be analyzed through the rules of the logical-mathematical structures that are “the basis of these figures,” i.e., they must be analyzed in the light of the proposals that must be formulated, regardless of their application in figures or use in constructions.

> That is why Hilbert emphasizes that, particularly in geometry, where mathematical notions are involved in figures, “a rigorous axiomatic discussion of its intuitive content is absolutely necessary.” Besides, once these structures are highlighted, and axioms are given, it is a matter of proceeding with a finite number deduction, that is, the deductions will consist of the application of procedures that have an end and effectively lead to a given result. In saying this, Hilbert has not yet realized that he is making a demonstration (deduction from theorems of axioms), a numerical procedure as we are used to in calculations of elementary arithmetic of integers (CHAUVE, 2006, p. 12).

From a philosophical point of view, the requirement of demonstrative rigor represents, for Hilbert, a general law of our understanding, i.e., all questions that arise from our understanding are likely to be resolved by it. In mathematics, this law means that, for any problem, we must be able to obtain the solution by demonstration or demonstrate the impossibility of the solution from the assumptions that are given or formulated. Hilbert (1899, apud CHAUVE, 2006, p. 12) says: “Any mathematical problem must be focused on a rigorous solution, either by a direct answer to the question or by demonstrating the impossibility of solving it, that is, the failure of any attempt of resolution.” The demand for rigor in reasoning is thus imposed on us by the conviction that any problem can be solved. This belief, which goes beyond the field of mathematics, expresses “a philosophical need.”

The author assures that to meet this requirement, it will be necessary not only to axiomatize mathematical theorems but also to reduce their demonstrative approaches to “finitist” procedures, as Hilbert says, i.e., a mode of thought considered fundamental and that must be the way that it works in every theory and every mathematical reasoning in the whole domain of demonstration. He states that the

> Finitist reasoning rests on a finite number of steps and finite data, for example, on specific numbers or finite collections of numbers, but it never introduces the consideration of the infinite totality, for example, that of the infinite set of numbers or the infinite set of decimal places of an irrational number. Such a procedure is comparable to the one applied in calculations when, for example, addition or multiplication operations are carried out. If we reduce the mathematical discourse to the steps of a finitist thought, any mathematical theory can become a system where the deduction of propositions will take the form of a calculation procedure (CHAUVE, 2006, p. 13).

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Is it mathematically possible? Hilbert (1899) thought so. Axiomization, in fact, has shown us that a theory can be presented as a demonstrative system in which theorems are deduced from axioms according to the pure rules of the logical-mathematical syntax of expressions. In this system, there are only successive sequences of propositions that consist of logical-mathematical signs - in other words, they are formulas - as if it were a type of algebra. A formula is a series of signs; a demonstration is a series of formulas, and the procedures to write these signals are codified. Hilbert (1925, apud CHAUVE, 2006, p. 14) summarizes this by explaining that instead of mathematical theories we have “a stock of formulas composed of mathematical and logical signals linked one after another according to defined rules.” In this system, a demonstration “constitutes a concrete object that can be visualized exactly like a number,” so that to reason and demonstrate it is enough to apply procedures comparable to those applied to the numbers in the calculations.

> By transforming mathematical theories into formal systems in which demonstrative approaches take the form of calculation procedures, Hilbert thinks that we can respond to the injunction dictated by the “philosophical need” to know that any problem can be solved. To be able to argue with all rigor the formalism that governs the demonstrations of a formal system, Hilbert demands that we stick to effective methods and finitist reasoning. This mathematical study of the formal systems of mathematics is called, for Hilbert, metamathematics, and, since it aims at the demonstrative possibilities of the systems, it will be a “demonstration theory” (CHAUVE, 2006, p. 14).

This is the task that the Göttingen school proposes to carry out from 1922 onwards and which is called “the Hilbert program.” It is a question of formalizing mathematical theories, of developing the theory of demonstration, that is, the metamathematical study of the properties of formal systems. At the Bologna International Congress, in 1928, Hilbert thought he had achieved the goal and explained the properties that, in his view, should have formal systems (in that sense, those properties are, therefore, standards for any formal system in which demonstrations are finitist and effective procedures). Those properties are consistency, integrity, and decidability. Chauve (2006, p. 15) presents it in the following way, which seems to clarify the idea better and which may be connected to Hilbertian formulations:

> - *Consistency* (called syntax): whatever formula A in the system, we never have A is demonstrable and not A is demonstrable. We can consider that it is in fact a norm, a property that we expect from a formal system and without which a system would cease to be one.
> - *Completeness* (called syntactic): we always have A demonstrable or A non-demonstrable (in this case, we say that A is refutable). If this condition is not met, we say that A is “undecided.”

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> - *Decidability*: it is always possible to "decide" whether A is demonstrable or whether A is not demonstrable. "Deciding" means: applying an effective procedure. If this condition is not satisfied, we will say that A is "undecidable."

The author asserts that if these three conditions are satisfied, we can completely dominate a system's field of demonstrations. From a formula of system A, we can decide whether it is demonstrable and, in this case, we will know that its negation is not, if it is not demonstrable, and in this case, we will know that its negation is (by completeness). We will have responded to the requirement of demonstrability that expresses a philosophical need:"we will know!" says Hilbert, and expects that the formal systems will give mathematics the chance to end ignorance.

Hilbert's conviction (1899, apud CHAUVE, 2006, p. 16) is that this way of thinking does not govern only the field of mathematical demonstrations; it governs all thought, when he says that "the fundamental philosophical conception [...] requires mathematics as well as all thought."

Kurt Gödel (1930) and Alan Turin (1936, apud CHAUVE, 2006) showed that the Hilbert program is unfeasible. Gödel shows that although the formal system of arithmetic is consistent, it is not complete. Turing, in turn, in his memoirs entitled *On Computable Numbers*, establishes that the "problem of Hilbert's decision has no solution" through effective methods and finitist reasoning. Chauve (2006) ensures that this means that there are formal systems that, although consistent, will not have the property of being complete or decidable. This does not exclude that we build formal systems that those properties will have (for example, that of "calculating propositions" in logic). Still, they will not be sufficient to formalize the arithmetic and all mathematical theories that we constructed based on arithmetic (the theory of real numbers, the theory of functions, and all analysis, consequently geometry). It invalidates the Hilbertian idea that mathematical theories come from fundamental and effective approaches to a finitist thought that could be established as a mathematically supreme authority, judging the demonstrative power of mathematics.

5 Conclusions

The mathematical demonstration can be seen as an argument by which someone is convinced or convinces others that something is true; therefore, it may seem difficult to go beyond the epistemic conversation about explanatory proof. However, although the content of any specific demonstration is the fruit of a person's epistemic work, it can be

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separated as an object independent of a particular mind. Others can read this demonstration and be convinced by it. What leads us to the question of knowing to show why a theorem is true is a characteristic of the very demonstration or communicative acts, either text or representations (BALACHEFF, 2019).

This must be compared with the criterion of recognizing the heuristic or epistemic character of an argument "[...] due to the existence of a theoretical organization of the field of knowledge and representations in which the argument occurs, or the absence of such a theoretical organization." (DUVAL, 1992, p. 51). In this perspective, "A heuristic argument requires the existence of a theoretical organization of the field of knowledge and representations in which the argument occurs" and "that is capable of understanding or producing a justification relationship between propositions that is of a deductive nature and not just of a semantic nature" (DUVAL, 1992, p. 52).

> Thus, the distinction between rhetorical and heuristic arguments is limited to the assessment of the epistemic value and the ontological value of the statements and their relationships. We can then advance that an argument will be admissible in the sense of mathematics if the epistemic value of its statements is conditioned by its ontological value; it is this criterion that will allow it to recognize the status of the proof in mathematics. The standardized structure of the demonstrations is the technical means of this assessment (BALACHEFF, 2019, p. 8).

The validation rules in mathematics stand out in relation to those of other disciplines because mathematicians do what they do because their objects are what they are at the time of their activities. The question of rigor is not abstract; it is a question whose possibility and nature of development depend both on epistemic conditions, in the Piagetian sense, and on technical means (representations and their treatments). (BALACHEFF, 2017)

The mutual dependence on conceptualization, representation systems, and validation systems makes it necessary to distinguish and characterize different types of proof, to be able to model possible developments and their conditions. The history of mathematics invites us to expand this perspective. If cognitive development is one of the determinants of the levels of validation – we have known this since Jean Piaget's work - , they are not the only ones. We must go beyond cognitive issues (Balacheff, 2019), taking into account, at least, the specific economy of validation situations and the state of knowledge.

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