Study

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ON THE CENTENARY OF ZYGMUNT JANISZEWSKI (1888-1920): IDEALS OF MATHEMATICAL PRACTICE AND THE CONSTITUTION OF CONTINUUM THEORY

SOBRE O CENTENÁRIO DE ZYGMUNT JANISZEWSKI (1888-1920): OS IDEAIS DA PRÁTICA MATEMÁTICA E A CONSTITUIÇÃO DA TEORIA DO CONTINUUM

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Abstract: This paper shows the importance of applying a certain approach to the history and philosophy of mathematical practice to the study of Zygmunt Janiszewski's contribution to the topological foundations of Continuum theory. In the first part, a biography of Janiszewski is presented. It emphasizes his role as one of the founders of the Polish School of Mathematics, and the social, political and military facets in which his intellectual character was revealed, as well as the values that guided his academic and scientific life. Kitcher's view of mathematical practice is then adopted to examine the philosophical conceptions and epistemological style of Janiszewski in relation to the construction of the formal axiomatic system of knowledge about the continua. Finally, it is shown the convenience of differentiating in Kitcher's approach, the methods, procedures, techniques and strategies of practice, and the aesthetic values of mathematics.

Keywords: Zygmunt Janiszewski; Continuum theory; Philosophy of mathematical practice; Polish school of mathematics.

Resumo: Este artigo mostra a importância de aplicar uma certa abordagem da história e da filosofia da prática matemática ao estudo da contribuição de Zygmunt Janiszewski para os fundamentos topológicos da Teoria do continuum. Na primeira parte, é apresentada uma biografia de Janiszewski. Ela enfatiza seu papel como um dos fundadores da Escola de Matemática da Polônia, e as facetas sociais, políticas e militares nas quais seu caráter intelectual foi revelado, assim como os valores que guiaram sua vida acadêmica e científica. A visão de Kitcher sobre a prática matemática é então adotada para examinar as concepções filosóficas e o estilo epistemológico de Janiszewski em relação à construção do sistema axiomático formal de conhecimento sobre o contínuo. Finalmente, mostra-se a conveniência de diferenciar na abordagem de Kitcher, os métodos, procedimentos, técnicas e estratégias da prática, e os valores estéticos da matemática.

Palavras-chave: Zygmunt Janiszewski; Teoria do Continuum; Filosofia da prática matemática; Escola de Matemática da Polônia.

1 Introduction

This year marks the 100th anniversary of the death of Polish mathematician Zygmunt Janiszewski, who died prematurely in early 1920 from the effects of the Spanish

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flu pandemic. The occasion invites us to review some aspects of his life and work, his vision as a teacher of the Polish School of Mathematics, his social and political commitment, his humanistic and philanthropic side, to focus on his education, his academic career and other aspects that shaped his vision and practice of mathematics.

Image recovered from *Portrety Uczonych Profesorowie Uniwersytetu Warszawskiego*. 1915–1945
See: (PIOTR; KAJETAN, 2016).

The first part of the article will refer to Janiszewski's biography, highlighting his stays abroad that influenced his ideas of positioning Polish mathematics as an international reference. This vision will consider the motivations of the Polish national context that encouraged him and a group of outstanding young mathematicians, with the idea of claiming a national mathematics with the consequent concentration of efforts and interests of new talents in a common branch of mathematics. A reference for this part is the work of (PRZENIOSŁO, 2011) in which Janiszewski's facets as a philanthropist, a military man and a mathematician serving the national cause of the formation of a mathematical school are analyzed. It also takes into account individuals and mathematical centers with which Janiszewski came into contact. These stays influenced the characteristics of his mathematical work, both in his doctoral thesis and in later publications, where he incorporates logic and formal approach into the point-set topology. This part will also consider the important historical episode of the creation of the journal *Fundamenta Mathematicae*, conceived by him and his colleagues Wracław Sierpiński and Stefan Mazurkiewicz within the shared vision of internationalizing Polish mathematics.

The second part studies Janiszewski's mathematical practice with reference to his major contribution, the foundations of the theory of continua, one of the most interesting

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chapters in the university teaching of general topology or point-set topology. As a continuum is a metric, compact and related space, its study implies previous familiarity with these concepts. Only after that it is possible to analyze the characteristic properties of mathematical objects such as arcs, Peano continua, chains of continua, decomposable and indecomposable continua or pseudo arcs. That is the standard presentation of the question in university texts, such as (CHRISTENSON, 1977).

Nevertheless, the explanation of the foundations of the theory of continua according to this logical order does not give account of background aspects for the history of mathematics and even for mathematical education. It remains unexplained what function this same logic of presentation fulfills in the system of knowledge. How a theorem or property of one of these objects is connected to a system of axioms and definitions with the purpose of fixing the logic of deductive chaining. If the philosopher or mathematics teacher wants to understand, for example, the role of continua in general topology, they cannot do it only by understanding the internal logic of its theoretical presentation.

Besides this, it is necessary to practice in the reconstruction of some significant moments of its history in order to become aware, even in a minimal way, of the heuristics of the process of this construction, the purposes to which it responded, the problems it came to solve, in other words, its reasons for being as such or such an object of the class of continuums. This reconstruction will also make evident how the current mathematical continuum drew boundaries with traditional views of the continuum, that is, how its properties acquired universal status by being established from a certain system of axioms.

That is precisely the general orientation of this work. We propose to examine some of the historical and philosophical transformations of mathematical practice that led in the early 20th century to the emergence of the formal axiomatic system of knowledge about continua and with which we are familiar in higher education. We adopt Kitcher's point of view in which the concept of mathematical practice is defined by five components: a language, a set of accepted statements, a set of accepted reasoning, a set of questions selected as important, and a set of meta-mathematical views that include not only the rules of proof and definition, but also "statements about the scope and structure of mathematics" (KITCHER, 1984). Additionally, we show the convenience of explicitly differentiating in the set of these components, the methods, procedures, techniques and strategies of practice, and the aesthetics and values of mathematics (ERNEST, 1998).We

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will illustrate this view of mathematical practice by studying Janiszewski's contribution to the topological foundations of continuum theory.

In the final part we study Janiszewski's philosophical conceptions and the epistemological style that identified his mathematical activity. To this end, we begin by reviewing his doctoral dissertation and his early research in order to appreciate the most original results of his contribution to the topological foundations of continua and locally connected spaces. In particular, his use of the set-theoretic approach to determine the intrinsic or topological definitions of sets such as arch and curve. Strict recourse to logic was then imposed both to establish the system of axioms from which the definitions were obtained and to express them at the highest level of generality. Janiszewski is distinguished by having introduced, in the Polish school and abroad, the use of symbolism and the procedures of formal logic in point-set topology. He believed that without the rigorous use of formal language it was not possible to establish the necessary hypotheses for the validity of reasoning. This will lead us to clarify the way in which Condorcet's new canon of rigor is adopted in his research.

Janiszewski's epistemological style is strongly oriented by ideals of abstraction and generality. We will study his novel attempt to extend the treatment of the properties of the sets of points to the incipient theory of Fréchet's spaces endowed with the topology of convergence, and to the metric spaces. Janiszewski's philosophical position of criticism of intuition as a criterion of truth is then presented, specifically in his research on irreducible continua, which contributed to clarifying ambiguous and controversial questions about curves and surfaces in mathematical practice at the beginning of the century. Finally, some observations are made about aesthetic ideals and values in connection with his search for forms of abstract thought in mathematics, a question that establishes a close relationship between Janiszewski and Poincaré.

2 Some Biographical Notes on Janiszewski

Zygmunt Janiszewski was a Polish mathematician, co-founder of the Polish School of Mathematics and *Fundamenta Mathematicae*, the first mathematical journal that specialized in a sub-discipline (set theory and its applications), with which an era of international splendor of Polish mathematics began in 1920.

He was born in Warsaw, his mother was Julia Szulc-Chojnicka and his father, the lawyer Czeslaw Janiszewski, graduated from Warsaw University and was employed as

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director of the Warsaw Municipal Credit Society. Following a school strike that accompanied the 1905 Revolution, the young Zygmunt moved to Lvov (a city that now belongs to Ukraine). In 1907 he began studying at the Faculty of Philosophy of the University of Zurich, but he did not find there favorable conditions to develop creative and independent mathematical thinking (PRZENIOSŁO, 2011). Despite this, he set out to create a mathematical circle with other Polish students in Zurich with whom he met weekly. It should be noted that although he did not continue his formal studies in philosophy, his interest in philosophy of mathematics and mathematical logic was evident from an early age. In this regard, (JADACKI; PAŚNICZEK, 2016) state that Janiszewski was one of the main initiators of cooperation between philosophers and mathematicians in Poland. They also comment that although Janiszewski studied abroad, he was in the sphere of influence of Kazimierz Twardowski, the father of the Polish School of Philosophy.

In the second semester of the first year of study he moved to the University of Göttingen, where he had contact with the mathematical community led by David Hilbert, characterized by cooperation and by stimulating young students. There Janiszewski had the opportunity to live in an environment conducive to mathematics. Despite this, Hilbert does not seem to have paid much attention to the interest that Janiszewski had been showing in toward the part of the topology that the French called *analysis situs*. In any case, it is quite possible that during his stay at this university Janiszewski became familiar with the new abstract approach of Hilbertian geometry, the set-theoretic methods and their applications to the analysis and generalization of the theory of curves, among other topics, through the works of Hilbert, Schönflies and Jordan, and that his research practice was impregnated with the atmosphere of axiomatic rigor and the mathematical formalism of Göttingen, as will be described later.

In the following years Janiszewski spent time in several major mathematical centers: between 1909 and 1910 in Munich, then again in Göttingen and finally in Paris in the academic year 1910-1911, where he completed his doctoral thesis *Sur les continus irréductibles entre deux points* (JANISZEWSKI, 1912) on 11 July 1911. In Paris, Janiszewski also perceived that there was a similar attitude to that of Göttingen, in regard to encouraging young talents. His teachers included Goursat, Hadamard, Lebesgue, Émile Picard and Poincaré. Let us add that in his thesis he also gives special recognition to Zoretti's 1908-1909 course. Lebesgue was his thesis advisor, and the examining

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committee consisted of Poincaré, Lebesgue, Borel and Fréchet (O'CONNOR; ROBERTSON, 2000).

Janiszewski's doctoral thesis is dedicated to Marc Sangnier, the creator of the *Le Sillon* movement, which was aimed at promoting a democratic and progressive Catholicism. These ideas shaped Janiszewski's social views without affecting his worldview. After defending his doctoral dissertation, Janiszewski left Catholicism, but his admiration for Sangnier remained forever. The idea of love of neighbor is a feature of Janiszewski's personality. The Polish mathematician Hugo Steinhaus remembers in this respect, that in the secondary school in Lvov, Janiszewski shared his clothes with people who needed them, and during his stay in Paris he ate in the dining room of the cooperative of the *sillonistes* without taking advantage of the benefit of the subsidy. Later, upon his return to Poland, he implemented the ideas of the *sillonistes* himself, allocating part of his resources to the education of talented and needy children.

In July 1911, Janiszewski attended the 11th Congress of Polish Scientists and Physicians in Krakow. It should be remembered that the first congress of mathematicians in Poland took place only in 1927. He participated in the section of exact sciences of this congress together with the most renowned professors of Polish universities, among them Stanislaw Zaremba and Kazimierz Zorawski of the Jagiellonian University in Krakow, and Jozef Puzyna and Wracław Sierpiński from Lvov. Puzyna and Sierpinski recommended him for a job at the University of Lvov in 1913, where he presented his thesis on aggregation (JANISZEWSKI, 1913) in which he formulated one of the most original results of his career. There were no common interests among the four mathematics professors attending the 1911 congress, as they worked in different specialties: Puzyna in analytical functions, Sierpiński in number theory and set theory, Zaremba in differential equations, and Zorawski in differential geometry. In this sense, the following quote from Sierpiński is interesting, as it prefigures the need to form a Polish School of Mathematics:

> After the congress I came to the conclusion that this was no good. There had been no collaboration, no mutual control. There were mathematicians known for their work abroad, but there was no Polish mathematics. My conclusion was that it would be better if a greater number of mathematicians worked in one area (DUDA, 1996, p. 482).

Janiszewski's novel idea of specializing a community in a single area of mathematics, through the creation of a specialized newspaper, pointed in the same

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direction. He communicated this idea to Fréchet, with whom he had maintained relations since his stay in Paris, in a letter dated February 29, 1912:

> I take this opportunity to consult your opinion on a question of a mathematical-social nature, which seems to me to be extremely important. It concerns the reform of contemporary mathematical periodicals and, to this end, the creation of a "model newspaper". Do you not notice the discomfort that comes from the profusion of mathematical periodicals (all with the same objective) and that it is so difficult to locate the literature of interest in such scattered conditions, especially when no good library is available? It seems to me that this great number of newspapers would be, on the contrary, of great comfort if each one had well delimited its specialty (for example, analytical theory of numbers, theory of substitutions...) (ARBOLEDA, 1982, p. 225).

At the end of World War I, the Mianowski Foundation, which sponsors research by Polish scientists, invited them to present their views on the needs of the various sciences in Poland. Janiszewski wrote a six-page article (JANISZEWSKI, 1917), which became a program for the next generation of mathematicians in his country. The article stated that Polish mathematicians can afford "not to be only receivers and consumers of foreign centers", but for Polish mathematics to position itself globally, scientific personnel must be concentrated in a specialized field of mathematics, and this field must be one in which Polish mathematicians have common interests and in which they have gained worldwide recognition. This field has to include set theory together with topology, and the foundations of mathematics together with mathematical logic.

Meanwhile, Janiszewski was lecturing in Warsaw on topology and philosophy of mathematics for students of the Society for Scientific Courses. In his free time, he organized mathematical and philosophical debates in his apartment with the assistance of young scientists.

Between 1911 and 1912 he was in Magdeburg, getting to know the mathematical and philosophical environment of the city. In April 1912 he participated in the International Congress of Mathematicians in Cambridge, giving a lecture on the concept of line and area, in which he showed that certain definitions used in geometry were insufficient and gave an example of a curve without arcs that is not a homeomorphic image of a segment of a line. Later on we will see the scientific and epistemological importance of this result in Janiszewski's mathematical practice. He spent several months of the following academic year in Bologna, then in Graz, Austria.

At the beginning of the war, he joined the legion that was fighting for Poland's independence. He participated as a soldier in the artillery in the winter campaign (1914-1915) in the Carpathians. A year later, along with several other legionnaires, he refused

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to swear allegiance to the Central Powers and had to take refuge, under the pseudonym of Zygmunt Wicherkiewicz, in Boiska near Zwoleń, and then in Ewin near Wloszczowa. In Ewin he ran a shelter for homeless children which he founded and supported with his own resources.

His nationalist spirit was not only philanthropic, it also became evident in his academic work, for example in the publication of the series *Guide for self-instruction*, in response to Russian restrictions on the teaching of mathematics in secondary schools (JANISZEWSKI, 1915). In the final part of the article we will return to the *Guide* in relation to Janiszewski's conceptions of mathematical practice.

In 1918, the University of Warsaw offered him a professorship in mathematics, and together with Mazurkiewicz and Sierpiński they set out to realize the dream of creating a School of Mathematics to promote the intellectual production of Polish mathematicians, with the establishment of the Seminary of Topology and the founding of *Fundamenta Mathematicae*. Kuratowski left the following testimony about the seminar:

> As early as 1917 [Janiszewski and Mazurkiewicz] were conducting a topology seminar, presumably the first in that new, exuberantly developing field. The meeting of that seminar, taken up to a large extent with sometimes quite vehement discussions between Janiszewski and Mazurkiewicz, were a real intellectual treat for the participants (KURATOWSKI, 1980, p. XXX).

Fundamenta Mathematicae, which, as mentioned above, was conceived as a journal specializing in set theory and its applications, a relatively new branch of mathematics and not yet consolidated at the time , came to public light in 1920, shortly after Janiszewski's death from the Spanish flu pandemic that had ravaged the world for the past two years. *Fundamenta* quickly gained acceptance from the global mathematical community. The first issue appeared under the direction of Mazurkiewicz and Sierpiński (after Janiszewski's death), with articles written only by Polish authors, and soon became the main vehicle for disseminating the most important results in its field of research.

It is rightly stated in (TAMARKIN, 1936) that the history of this journal became the history of modern function and set theory. *Fundamenta* allowed the Warsaw school to show the world a great example of mathematical growth from a peripheral situation. Janiszewski's great vision became a reality within a few years, as Warsaw became an internationally renowned center for set theory, producing renowned specialists who were important not only in the Polish or European arena, but who also gained a high degree of recognition at universities in the United States. This model of specialized journal was

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replicated in Poland in journals such as Studia Mathematica, devoted to function theory since 1929, and Acta Arithmetica, devoted to number theory since 1935.

As a remarkable fact, we should point out that on the Genealogy Project website Janiszewski appears as the director of a doctoral thesis, that of Kazimierz Kuratowski. From his death in 1920 Mazurkiewicz took over the direction until its completion in 1921. The thesis consisted of two parts: the first one was about an axiomatic construction of topology through cloistered axioms (reproduced with some modifications in volume 3 of *Fundamenta* (KURATOWSKI, 1922); this part corresponds to Mazurkiewicz's influence. The second part deals with the irreducible continuum between two points, a subject derived from Janiszewski's thesis. It was also published in volume 3 and other extensions of it appeared in a second part in volume 10 (KURATOWSKI, 1922a).

To conclude this biographical part, it is important to present the testimonies of Kuratowski and Knaster, two of Janiszewski's students, which summarize the intellectual, social, political and cultural characteristics of his thought, as well as his commitment to persevere at all costs in his ideals, and the search for an explanation of problems through pure mathematics:

> Janiszewski was an unusual personality, combining great creative talent, organizational talent, faith in the scholar's mission, ardent patriotism, a noble character and kind-heartedness. Receptive to ideas of progress and social justice, he underwent deep ideological changes. [He] "was turning more and more to the left in politics" (KURATOWSKI, 1980).
>
> [Janiszewski] donated for public education all the money he received for scientific prizes and an inheritance from his father. Before he died he willed his possessions for social works, his body for medical research, and his cranium for craniological study, desiring to be "useful after his death" (KNASTER,1970-1980).

3 Janiszewski's Philosophical Ideas and Epistemological Style in His Work on the Topological Theory of the Continuum

Besides being the founder, together with Sierpiński and Mazurkiewicz, of the Polish school of mathematics, Janiszewski is known in the history of mathematics for his contribution to the topological foundations of the theory of continuum. Apart from the originality of his results, his contribution is also associated with the remarkable fact of having introduced for the first time in this field a new point of view in the symbolic treatment of mathematical objects and in the formulation of their properties through the algebra of logic. We will quickly review his mathematical contribution as a reference

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framework to comment on the philosophical and epistemological ideals that mobilized his research practice.

In his biographical note on Janiszewski, Knaster states what he considers to be his three most original theorems (KNASTER, 1970-1990). For his part, Kuratowski endorses the notoriety of these same results in volume II of his *Topologie* with the first rigorous and formal exposition of the advances in research on continua and locally connected spaces, until the 1930s (KURATOWSKI, 1968). It should be remembered that the manuscript of this important work was already finished in 1939; the delay in its publication has to do with the paralysis of all academic activities due to the war and the German occupation of Poland (ENGELKING, 1998, p. 434).

The first original theorem of Janiszewski according to Knaster, in the equivalent formulation of (KURATOWSKI, 1968, p. 172) where it appears as the first theorem in the exposition of the chapter on related spaces, is the following:

1. If A is a closed proper subset of the continuum X and if C is a component of A, then

$$C \cap \overline{X - A} \neq 0, \;\; i,e. \quad C \cap \mathrm{Fr}\,(A) \neq 0.$$

Kuratowski gives as a reference for this theorem, which designates a characteristic property of closed subsets of continua, the aggregation thesis of Janiszewski (1913). Janiszewski's early interest in the study of topological properties of the closure is evident in several parts of his thesis (JANISZEWSKI, 1911). It is possible to affirm that this was a strong motivation in Kuratowski's own preferences for choosing the topology of closure to determine the *intrinsic geometry* of the abstract spaces. According to Engelking's testimony:

> As Professor Kuratowski emphasized, Janiszewski was the first topologist to understand the predominance of the closure over the set of accumulation points. Following his line, the young Kuratowski, a friend and the most talented student of Janiszewski, devised the famous axiomatics in terms of the closure operator, axiomatics that allows an exact translation to the open set (or neighborhood) approach (ENGELKING, 1998, p. 445).

The second most important theorem of Janiszewski in Knaster's view is the following (KURATOWSKI, 1968, p. 214):

2. If a continuum is irreducible between two points and does not contain subcontinua which are non-dense on it, then it is an arc.

Kuratowski recalls that this *Theorem of Janiszewski* was formulated in (JANISZEWSKI, 1911). In fact, this theorem represents the concretion of the purpose that Janiszewski stated in the Introduction of the thesis, to find an intrinsic (topological) characterization of the notion of arc. As we will see later, this approach of defining objects

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by their characteristic properties constitutes one of the central aspects of Janiszewski's rigorous epistemological style. To the extent that the determination of the concept of arc incorporates various topological notions of the continua, Kuratowski shows in his *Topologie II* the outstanding role that this theorem plays in the study of closed subsets of irreducible spaces (JANISZEWSKI, 1911, p. 214).

Kuratowski also establishes correlations between the theorem and previous concepts of the arc such as homeomorphic space to the interval [0,1], simple closed curve (a homeomorphic space to the unit circle), and the notion of "separator": each interval contains exactly two points that do not separate it, called end-points of the arc. Finally, he compares the previous topological characterization of an arc with the following one that translates the interest that Janiszewski expressed since the beginning of his research to extend the topological properties of the sets of points of the plane and E^n to Fréchet's metric spaces or V classes: “If every point x of a metric continuum X, with exception of two points a y b, is a separator, then X is an arc” (KURATOWSKI, 1968, p. 179).

The third original theorem of Janiszewski according to Knaster corresponds to the property (J) that according to Kuratowski characterizes the *Janiszewski spaces* (KURATOWSKI, 1968, p. 505):

3. X is said to be a *Janiszewski space* if X is a locally connected continuum having the following property:

(J) If C_o and C_1 are two continua whose intersection $C_0 \cap C_1$ is not connected, the union $C_0 \cup C_1$ is a cut of the space.

Kuratowski recalls (1968, p. 506) that the sphere of dimension 2 is a Janiszewski space. Both properties were introduced by Janiszewski in his habilitation thesis (Janiszewski, 1913) by studying the disconnection of the plane by continua, one of the key problems for the formalization of the plane topology as it becomes evident in *Topologie II*. Kuratowski shows the equivalence of the previous property (J) with the following one (*loc. cit.*, Theorem 7, p.507):

> (J') X is a locally connected continuum. Let A and B be two closed sets of X. If none of these sets separe X between the points x and y, and if $A \cap B$ is connected, then $A \cup B$ does not separate X between x and y.

According to Engelking, in the case where X is the sphere of dimension 2 the property (J') is the so-called *First Janiszewski Theorem* and (J) the *Second Janiszewski Theorem.* This is why Kuratowski calls Janiszewski space a locally connected continuum that satisfies (J) (ENGELKING, 1998, p. 438).

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Having reviewed the three most important findings of Janiszewski's continuum research, we now turn to the philosophical and epistemological insights that influenced its development.

In the Introduction to the thesis (JANISZEWSKI, 1912a) several ideas are put forward about the method of his investigations, which, as we shall see, follows the new epistemological canon of rigor and formalism in early 20th century mathematics. He comments that his first problem consisted in using the point-set approach to make the axiomatic study of certain sets of points, such as arc and curve, which had not been defined precisely until then. To remove the ambiguities and confusion between the two concepts, Janiszewski opted for intrinsic definitions that consisted in "characterizing the sets only by the [topological] properties of the points of these sets... [without appealing to] "the relations between the points of the set and the rest of the space" (JANISZEWSKI, 1912a, p. 80). Janiszewski had observed that constructive definitions, such as an arc by a homeomorphism of the interval [0,1] in the space, were, so to speak, insufficient. He could even be aware that it was possible to construct a curve that did not contain arcs, as he would show in his communication of the year following the Cambridge Congress (JANISZEWSKI, 1912b).

The point of view of the intrinsic definition led to the recourse to logic, since it was necessary to establish the minimum axioms from which the properties of the defined objects were deduced. It was also a means of orienting research towards the search for the highest possible generality in the treatment of problems. In fact, the rigorous use of formal language allowed him to recognize the "hypotheses needed for the validity of the reasoning", as well as "to obtain valid results for the Euclidean space of any number of dimensions, and for abstract classes of other spaces (JANISZEWSKI, 1912b, p. 80). We will now comment on the influence of both ideas on Janiszewski's mathematical practice.

Janiszewski's style is distinguished above all by the strict application of symbolism and the rules of formal logic in theoretical exposition. His basic argument is the search for rigor in the formulation of ideas and the simplicity that this entails. For example, in chapter I with the preliminaries of the thesis, Janiszewski employs the symbolism and procedures of formal logic to designate fundamental operations and identities of set theory. This new language allows him to define basic concepts and properties of conjunctive topology as closed, perfect, dense, boundary point, derivative, etc. It then shows the convenience and economy of thought that results from employing this topological "logistics" in the study and characterization of continua.

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The fertility of this logical treatment is also indicated in Note I of the thesis. By introducing symbolic calculus to the study of the properties of irreducible continua in n −dimensional Euclidean spaces, Janiszewski was able to recognize that it was possible to generalize almost all these properties to the abstract spaces that Fréchet had recently introduced in his thesis (Fréchet, 1906). Namely, in the L classes with the topology of convergence and in the V classes or metric spaces. Before exploring in Couturat the origins of the logical treatment employed by Janiszewski in his investigations into the continuum, it is worthwhile to consider the influence that the reading of the thesis and other works by Fréchet on the topology of sets of points in abstract spaces might have had on his ideas.

At the end of the Introduction he states that the domain of validity of his results on the continua is not Euclidean geometry (JANISZEWSKI, 1912a, p. 83). He then explains in Note 1 that the domain of his research is Euclidean n −dimensional space, but he clarifies that the points in this space are of any nature and not n −uplas of real numbers. He notes in this respect that the domain of n −uplas "is not Euclidean space, but *a* certain space, *a* certain interpretation of Euclidean space" (sic; JANISZEWSKI, 1912a, p. 153).

He acknowledges having adopted Hilbert's geometry approach to the plane, which consists of building a geometry that contains all geometries as particular cases. He emphasizes the fact that neither in his reasoning nor in the axiomatic-deductive presentation of his theorems do the geometrical properties of space enter into play: "Indeed, I do not use there any axiom or elementary geometrical notion as a line or plane" (JANISZEWSKI, 1912a, p. 154). Except for the extension of the Euclidean distance to the abstract spaces in which the notion of Fréchet's generalized limit has been defined, that is to say, the spaces endowed with the topology of convergence of successions of points (JANISZEWSKI, 1912a, p. 154).

He also draws attention in Note 1 to the advantages of studying properties of sets of points in Fréchet's abstract spaces or L classes, since the same properties of these sets can then be applied to sets of spaces of a particular nature, provided that the notion of limit has been defined in them. This is precisely the abstract and general point of view that Janiszewski applies to the study of the fundamental topological properties of continua. This approach allows him, in particular, to analyze the notion of arc simple in chapter III of his thesis, and to obtain one of his most original results, the *Theorem of Janiszewski* referred to above.

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This axiomatic study of the abstract space, not only has an interest in itself, but "allows to pass from the properties of the sets of points in a certain space to the properties of the sets of points considered in another space, as the study of abstract sets allows to pass from sets of points to sets composed of other elements" (JANISZEWSKI, 1912a, p. 155). Janiszewski thus places himself in the epistemological perspective of the beginning of the century consisting of adopting the algebraic ideal in the topological study of mathematical objects in abstract spaces; what was then known as the study of the *intrinsic geometry* of such spaces.

By extending the notion of the limit of defined sequences in specific sets to a set of elements of any nature, Fréchet says in his thesis that he has proceeded in accordance with recent developments in abstract group theory (FRÉCHET, 1906, p. 4-5) (ARBOLEDA, 2017, p. 530). Previously, research in this field had been carried out separately on theories of groups of movements, substitutions, transformations, etc., in which the definition of the *mode of composition* varied from one theory to another. But the experience with different systems of axioms in specific groups gradually led to the conviction that:

> It was only possible to arrive at a common theory by abstaining from giving a general definition of the mode of composition, and seeking the conditions being common to the particular definitions, and taking into account only those that were independent of the nature of the elements considered (FRÉCHET, 1906, p. 5).

The same idea can be found in the Notice about his mathematical works, where Fréchet recalls the structural approach used in his works on General Topology, which consisted in "avoiding the fastidious repetition of theories and demonstrations that were essentially the same, although relating to object domains of a diverse nature: numbers, curves, surfaces, functions, series, groups, random variables, etc., etc" (FRÉCHET, 1933). Although Janiszewski agreed to adopt this structural approach, he was perhaps able to distance himself from Fréchet's lack of rigor in applying the axiomatic-deductive method. Janiszewski criticizes in the Introduction the habit of extending without demonstration to Euclidean spaces and to abstract spaces propositions already known in the line and plane: "my reservation is explained because several allegedly evident propositions finally turned out to be false" (FRÉCHET, 1933, p. 82). It is possible to imagine that he recognized this informal style of exposition in Fréchet's thesis. Indeed, although Fréchet exposes his ideas roughly according to the axiomatic method, he uses a

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hybrid narrative in which natural language expressions often artificially replace the use of point-set language and logical symbolism.

But it is above all Fréchet's expository style that lacks rigor in the demonstration of some of the theorems, particularly when they are generalizations to abstract spaces of propositions previously demonstrated in domains of objects of a specific nature. In these cases, Fréchet is reduced to pointing out particular aspects of the proof or to supporting the validity of the propositions in the evidence of their applications. This seems to be related to his philosophical conceptions on the constitution of abstract objects from the experience of the concrete through an inductive synthesis procedure (ARBOLEDA; RECALDE, 2003, p. 255-265). One of the theorems of Fréchet's thesis, whose deficient demonstration must have attracted Janiszewski's attention, states that "being U a continuous [functional] operation defined in a continuous set E, there exists at least one element of E where U takes all the values comprised between any two values taken by U". Fréchet comments that this is an extension of the theorem that Cesare Arzelà had previously formulated in the restricted domain of line functions, and he merely points out some conditions on the E continuum that would allow the same scheme of proof to be adopted in the more general case (FRÉCHET, 1906, p. 9).

For his part, Janiszewski considers that without the rigorous use of formal language it is not possible to establish the necessary hypotheses for the validity of reasoning. This demand for rigor arises when determining the system of axioms that define the characteristic properties of objects, and when, from this same system, the propositions about the relationships between such objects are deduced as truths. As already mentioned, Janiszewski recognizes in his thesis that his research is framed in the new canon of rigor of *L'Algèbre de la logique* (COUTURAT, 1905), one of the precursors of symbolic logic in France and Europe. In the preface to the English translation, Jourdain highlights the following aspects of the work that allow us to better understand the ideals of rigor and in general the epistemological style of Janiszewski's mathematical practice (COUTURAT, 1914):

- The primary significance of a symbolic calculus seems to lie in the economy of mental effort which it brings about, and is the reason for the characteristic power and rapid development of mathematical knowledge (p. i).
- The objects of a complete logical symbolism are: firstly, to avoid this disadvantage by providing an *ideography* [Frege], in which the signs represent ideas and the relations between them *directly* (without the intermediary of words), and secondly, so to manage that, from given premises, we can, in this ideography, draw all the logical conclusions they imply by means of rules of transformation of formulas analogous to those

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of algebra, -in fact, in which we can replace (substitute) reasoning by the almost mechanical process of calculation. This second requirement is the requirement of a *calculus ratiocinator* [Leibniz] (p. iii).

- The present work deals with the calculus ratiocinator aspect, and shows, in an admirably succinct form, the beauty, symmetry and simplicity of the calculus of logic regarded as an algebra (p. iv).

The rigorous logical presentation of results in the style of Janiszewski's thesis will be one of the distinguishing features of the works of the 1920s in the theory of continua (Mazurkiewicz, Knaster and Kuratowski) and, in general, in the publications of the Polish School of Mathematics. Let us note in particular that Kuratowski's *Topologie I* (1933-1958-1966) begins with a long introduction in which the notations and elementary theorems of set theory and the algebra of logic are exposed. The latter is used especially in problems of function theory "in which the use of logical notations is very naturally imposed and allows simplifying the reasoning". But Kuratowski's logicist and formalist style is not only evident in the presentation of this work; it is also evident in his 1920 thesis which, as we know, was largely oriented by Janiszewski. At the beginning of the work (KURATOWSKI, 1922), it is noted that in the presentation of the basic notions of the topology of closure in the $n-$ dimensional Euclidean space, he will proceed in an axiomatic way.

Indeed, all topological reasoning is deduced from the system of axioms that define the closure operator, through the application of the point-set formulas and properties and in accordance with the logicist approach of Couturat. The rigorous adherence to logical calculus leads Kuratowski (1922) to synthesize in a table the formulas and relations used in the characterization of the topology of closure. This table will be key for the logical analysis of the system of axioms of the closure, specifically to demonstrate its independence on the one hand, and its equivalence with the system of axioms of the derived set operation, another of the resources then in vogue for studying the topology of abstract space. Let us note in passing that contrary to what might be expected, the reference to *L'Algèbre de la Logique* in (KURATOWSKI, 1922) is not to its Russian or Polish translations already available at that time, but to the original in French (COUTURAT, 1905).

Another interesting topic in the Introduction to Janiszewski's thesis, as it reveals his philosophical approach to mathematical practice, is his critique of intuition as a criterion of truth. He states that he has arrived at this conviction through his own experience, particularly in the study of the irreducible continuum, having found that many propositions previously accepted as true later turned out to be false. Janiszewski is no

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doubt thinking of Brouwer's (1910) counterexamples, pointing out the existence of errors in certain theorems of Zoretti on the decomposition of irreducible continua. Zoretti (1909) introduced these new objects by studying different mathematical conceptions of the notion of curve, in particular those of Cantor and Jordan. Let us note in passing that Janiszewski's orientation in this direction had to do with at least two circumstances: the research problem proposed to him by Lebesgue, his thesis tutor, and his attendance at the 1908-1909 course at the *Collège de France* (Paris) in which Zoretti presented his results on irreducible continua. This problem was certainly related to one of the above-mentioned errors by Zoretti that led to Brouwer's (pathological) counterexample.

In his remarkable article *History of continuum theory*, Charatonik summarizes the subject in the following terms (CHARATONIK, 1998, p. 707 and 717): Initially, a continuum was understood as a connected, closed and bounded (sometimes not necessarily bounded) subset of a Euclidian space. In the first chapter of his thesis Janiszewski defines a continuum as a closed and well-chained set that is not reduced to a point (JANISZEWSKI, 1912, p. 84). A continuum is irreducible between two of its points if no subcontinuum contains those points. One of Zoretti's false theorems assumed that the outer boundary of a domain can be decomposed into two subcontinua with only two points in common. In other words, Zoretti considered that an irreducible continuum generalizes a simple arc, and from that he concluded that from any irreducible continuum a linear order can be obtained. On the other hand, in Brouwer's counterexample there was an indecomposable continuum whose construction allowed the description of a common boundary of a finite number (greater than two) or even a numerable one of many domains. Janiszewski's personal research on these and other properties of the irreducible continuum shed light on ambiguous and controversial questions, and allowed him to arrive at original results such as the theorems already mentioned in the foundation of the emerging theory of the continuum.

It is interesting to recall in this regard Knaster's view that his natural mathematical insight, coupled with a taste for applying logic to reasoning, enabled Janiszewski to recognize structural deficiencies and ambiguities in basic geometric concepts such as line and surface. In his paper at the Cambridge Congress in 1912, devoted precisely to investigating the topological properties that allow these concepts to be characterized in a rigorous and general way, he presents the schematic procedure for constructing a curve (a dimension 1 continuum), which does not contain an arc (a homeomorphic image of an

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interval). This would be the first curve without arcs in the history of mathematics (KNASTER, 1960, p. 2).

The paradoxical nature of this curve revealed the limitations of the mathematical intuitive view of the continuum that was predominant at the time, for which the concepts of curve and arc were inseparably interrelated. This mathematical intuition was inherited from the traditional experience of the indivisible continuum, limited to the treatment of discrete quantities and their construction from processes of endless division. Therefore, in the field of this intuition it was not possible to imagine the new objects that Janiszewski was introducing into the emerging theory of the continuum. This required a new intuition with a *meta-geometric* mentality. An approach capable of capturing the descriptive definitions of the intrinsic properties of the continuum was indispensable, through the use of the point-set approach and new topological concepts of abstract sets of points such as connectedness and compactness. This approach would give rise to new forms of intuition associated with the epistemological character of the new theoretical field.

In a recent study on the problem of mathematical intuition in the works of Polish mathematicians of the beginning of the century, including Janiszewski, Wójcik proposes to interpret their conceptions about the process of emergence of the theory and the concomitant forms of intuition, through the following general scheme (WÓJCIK, 2019, p. 169):

- A certain mathematical theory is assumed as the basis for the construction and definition of new objects; in the case of Janiszewski and the emerging theory of continua, this is the nascent theory of sets of points.
- A system of axioms is determined with the minimum necessary properties to define objects, in such a way that *good* intuitions are preserved and *improper* ones are abandoned.
- The tools of mathematical logic (logical analysis) are applied to the study of the above questions.

Claude Chevalley, one of the founders of the famous Bourbaki School of Mathematics, refers to several aspects of the *epistemological style* of mathematics at the beginning of the century (CHEVALLEY, 1935). He specifies that this new style marked a break with the nineteenth-century style program known as the *arithmetization of analysis*, in terms of its purpose of constructing a unitary mathematics from real numbers. He warns that the most important thing for a new geometry is not the nature of the points in space. Constructive definitions from (real) numbers provide the geometry with a chaos of points with no relation to a structure. Modern mathematics is not constructive in this sense but descriptive: it is a matter of defining objects by

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comprehension, i.e. by their characteristic properties, and not by extension. In the manner of Hilbert in his axiomatic plane geometry, or of Fréchet in the topology of sets of points in abstract spaces:

> (...) instead of defining the points, lines, and so on, from other notions, and then deducing from them their properties, [in these new geometries] the nature of these objects is left completely undetermined, being satisfied with giving their descriptive definition consisting of the statement of a certain number of their fundamental properties called axioms (CHEVALLEY, 1935, p. 380).

The use of logic and the axiomatic method was a distinctive feature of the new epistemological style. The rigor and conceptual clarity of the operations involved in the calculus were based on the choice of the system of axioms with the strictly indispensable properties to deduce the results from them. The validation of the results by a minimal demonstration was a way of confirming their relevance in the respective domain of mathematics. Otherwise, it allowed the rejection of methods that could possibly introduce useless hypotheses. "The care taken in the exact adaptation of methods, while giving them a precise meaning, rewards the search for elegance in demonstrations, something that was neglected by the geometricians of the preceding school" (CHEVALLEY, 1935, p. 382).

Finally, we will make some brief observations on Janiszewski's ideals and aesthetic values in connection with abstract forms of thought in mathematics. We will make use of his introduction to *Guide for self-instruction* (JANISZEWSKI, 1915), in which he sets out his views on the nature of mathematics, its purposes, its philosophy, its relations with other sciences, in particular logic, and its importance for education in general. In questions of the truth and beauty of mathematics, his main inspiration was Poincaré. As summarized by Wójcik, Janiszewski and many other Polish mathematicians shared Poincaré's position on these matters. For them:

> (…) the main goal of science is to explore the hidden beauty, harmony of the world, which is accompanied by recognizing and constructing the beauty of simple facts and mathematical formulas. It has its groundwork in the instinctive desire for beauty which strengthens the selfless search for the truth (WÓJCIK, 2019, p. 163).

In Janiszewski's Introduction he takes up several of these ideas from Poincaré, relying particularly on *Notice sur Halphen* (POINCARÉ, 1890) and the Polish translation of *Valeur de la science* (POINCARÉ, 1905). On the question of the common people about the raison d'être of the abstract constructions of mathematics, his answer is to claim first the taste of mathematics for its intrinsic interest and then for its possible applications. This ideal is often expressed in a quote that his colleague Hugo Steinhaus attributes to Janiszewski: "I do mathematics to see how far I can go on the path of pure reasoning"

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(URBANEK, 1919). Returning to his reading of Poincaré in the *Guide*, let us observe that Janiszewski paraphrases at a certain point and then quotes the following text in which Poincaré exposes his ideas on the aesthetic value of these constructions, explaining the characteristics of Halphen's work in algebraic geometry:

> The scholar worthy of the name, especially the geometrician, experiences the same impression as the artist in front of his work; his enjoyment is as great and of the same nature. (...) If we work, it is less to obtain these positive results, to which the common man believes we are only attached, than to feel this aesthetic emotion and communicate it to those who are capable of experiencing it (POINCARÉ, 1890, p. 143) (JANISZEWSKI, 1915, p. 14).

Janiszewski adds that this aesthetic emotion can only be truly appreciated in the practice of mathematics. The supporting quote is now the famous letter to Sophie Germain, in which Carl Friedrich Gauss legitimizes the work of women in the face of the prejudices of the time (JANISZEWSKI, 1915, p. 15): "The taste for the abstract sciences in general and, above all, for the mysteries of numbers, is very rare: this is not surprising, since the charms of this sublime science in all their beauty reveal themselves only to those who have the courage to fathom them." However, those who cannot perceive this beauty for themselves must be content to recognize it by comparison with the beauty produced by music and architecture.

Returning to the *Notice sur Halphen*, it is possible to speculate whether, beyond sharing the canon of beauty referred to by Poincaré, Janiszewski (the mathematician, the military man, the patriot) was attracted to the figure of the renowned French geometrician, author of interesting results on algebraic curve singularities, who additionally earned honors for his military performances as an artillery lieutenant. All the more so because Hermite (quoted by Poincaré) takes Halphen as an example of the virtues of abstract mathematical training applied to war operations:

> there can be no doubt that mathematical studies help to form that faculty of abstraction indispensable to the leader to form an inner image, an image of the action by which he directs himself, forgetting the danger, in the tumult and darkness of combat (POINCARÉ, 1890, p. 138).

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