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HEGEL, PEIRCE AND US

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M. Otte[1]

Abstract: Historically, our theme is situated within the triangle of three of Kant's students: Hegel (1770-1831), Bolzano (1781-1848) and Peirce (1839-1914). All three wanted to change Kant's strict separation of philosophy and science by developing a new conception of logic. Bolzano inaugurated the so-called *linguistic turn* of philosophy which became the guiding principle of all analytical philosophy (Dummett, 2014) and he opposed Hegel's unity of concept and object of knowledge. Charles Peirce took a middle position, a position that is expressed in his so-called *Pragmatic Maxim* (Peirce, CP 5.3). Taken together we might say that a universal *principle of complementarity* of meaning and reference, or of meaning and information (in the sense of Shannon) finds its origin in Post-Kantian philosophy. We encounter here the very same approach of principled thinking endorsed by Einstein in physics (special theory of relativity) or by the formal axiomatic approach in mathematics (Hilbert)!

Key Words: Bolzano, Hegel, Peirce; Complementarity of sense and reference; Geometry from Euclid to Einstein; Hilbert.

Resumo: Historicamente, nosso tema se situa dentro do triângulo de três alunos de Kant: Hegel (1770-1831), Bolzano (1781-1848) e Peirce (1839-1914). Todos os três queriam mudar a separação estrita de Kant entre filosofia e ciência, desenvolvendo uma nova concepção de lógica. Bolzano inaugurou a chamada virada linguística da filosofia, que se tornou o princípio orientador de toda filosofia analítica (Dummett, 2014) e se opôs à unidade de conceito e objeto de conhecimento de Hegel. Charles Peirce assumiu uma posição intermediária, posição que se expressa em sua assim chamada Máxima Pragmática (Peirce, CP 5.3). Tomados em conjunto, podemos dizer que um princípio universal de complementaridade de significado e referência, ou de significado e informação (no sentido de Shannon) encontra sua origem na filosofia pós-kantiana. Encontramos aqui a mesma abordagem do pensamento baseado em princípios endossada por Einstein na física (teoria da relatividade especial) ou pela abordagem axiomática formal na matemática (Hilbert)!

Palavras-chave: Bolzano, Hegel, Peirce; Complementaridade de sentido e referência; Geometria de Euclides a Einstein; Hilbert.

1 Introduction

Historically, our theme is situated within the triangle of three of Kant's students: Hegel (1770-1831), Bolzano (1781-1848) and Peirce (1839-1914). With his so-called *Copernican Revolution of Epistemology*, Kant created a new stage on which all further philosophical disputes then took place. For Kant, all knowledge was bound to a human subject. And it was only in this relationship to the human subject, that the order of knowledge could be represented. Kant's epistemology was shaped by the idea that the subject realized its essence through his activity in the objective world. This view became

[1] PhD in Mathematics from the University of Goettingen and Munster University (Alemanha). Visiting foreign Teacher at the Federal University of Mato Grosso (UFMT). E-mail: michaelontra@aol.com

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radicalized by Fichte in terms of his famous *I am I*, and became the starting point of German idealistic philosophy of Schelling and Hegel. D. R. Lachterman reformulates the basic idea as follows:

> The constructivist project, rooted in Descartes' geometry and exfoliated in Kant's critical enterprise, took its bearings from the desire to master and possess nature, where nature was understood as the locus of apparently ineliminable or intractable otherness. Mind could aspire to master its other … by externalizing itself in a construction carrying the clear marks of inward and deliberate origin (LACHTERMAN,1989, p. 23).

While for Kant knowledge essentially meant the construction of concepts in intuition, Bolzano relied on language and the sentence. In his criticism of the skepticism of *Sextus Empiricus* – Bolzano grounds his entire *Logic* or *Doctrine of Science* (*Wissenschaftslehre*) on a single sentence. Namely on the sentence: *There are true sentences.* This sentence cannot be refuted, because the negation of it, "*There are no true sentences"* is itself a sentence and as such it is either true or false. Assuming that it is true, leads to contradiction.

In difference to the classical skepticism of *Sextus Empiricus* Bolzano does not need to know, whether a certain sentence is true nor to be able to distinguish between truth and falsehood in general (BOLZANO, 1981, §25). In this way, the ontology is based on the principle of non-contradiction, and truth and knowledge are separated. As a rule, we cannot even know whether a certain sentence is true or not. Bolzano is very well aware of the novelty of his "linguistic turn" (BOLZANO, 1981, §33).

Communication and meaning come to dominate over existence and reference. Few things, Coffa writes, commenting on Bolzanos philosophy, "have proved more difficult to achieve in the development of semantics than recognition of the fact that between our subjective representations and the world of things we talk about, there is a third element: what we say" (COFFA, 1993, p.76). However, semantics can be understood in two different ways, namely as the branch of linguistics that deals with the study of meaning and communication, or, secondly, as the study of the relationships between signs or symbols and what they represent. Bolzano adheres to the first view; Peirce endorses the second.

Hegel essentially tied in with Kant's so-called *Copernican Revolution in Epistemology*, whereby the object of knowledge is reduced or traced back to the epistemic subject. But Hegel emphasized the primacy of conceptual thinking, while for Kant, above all, (pure) intuition determines the foundation of knowledge. The adherence to the importance of intuition marks the basic principle of Kantian epistemology. For Kant or

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Peirce, mathematics and natural science are therefore much more important than for Bolzano and Hegel. As Kant said: "Philosophical cognition is the cognition of reason by means of concepts, mathematical cognition is the cognition by means of the construction of concepts in intuition" (KANT, CpR, B742).

Hegel made the evolution of the complementarity of concept and object the touchstone of his evolutionary view of knowledge. And he criticizes Kant in particular for the often-missing connection between concept and object. In the introduction to his *Phenomenology of Mind* Hegel writes:

“Suppose we call knowledge the *concept*, and call the essence or truth *being* or the *object*, then the examination consists in seeing whether the notion corresponds with the object. But if we call the essence of the object, or what it is in itself, the concept, and, on the other side, understand by *object* the notion qua object, i.e. the way the notion is for another, then the examination consists in our seeing whether the object corresponds to its concept. It is clear, of course, that both of these processes are the same” (HEGEL, 1952, p.71).

This equality of relationships must be understood from the point of view of knowledge development. It is something revealing itself only in the course of the evolution of knowledge. It seems obvious, for example, that a collection of individuals of a species is possibly the same as a collection of the perspectives on reality. Berkeley had said this already and had explained it in semiotic terms, that is, as a relation between signs and things. Peirce had rephrased Berkeley by saying: “*cognizability* and *being* are not merely metaphysically the same, but are synonymous terms” (PEIRCE, CP 5.257).

And at some other occasion Peirce had explained the evolutionary process of cognition as follows:

> At first sight it seems no doubt a paradoxical statement that, *the object of final belief which exists only in consequence of the belief, should itself produce the belief*; but there have been a great many instances in which we have adopted a conception of existence similar to this. The object of the belief exists it is true, only because the belief exists; but this is not the same as to say that it begins to exist first when the belief begins to exist. We say that a diamond is hard. And in what does the hardness consist? It consists merely in the fact that nothing will scratch it; therefore, its hardness is entirely constituted by the fact of something rubbing against it with force without scratching it” (PEIRCE, CP 7.340).

Hegelian philosophy and the associated dialectical method served to introduce the point of view of development and evolution into philosophy and science. A very different motive ruled over the efforts of Bolzano, Frege, Weierstrass, Dedekind and others. The logical ambiguities of common language and problems of knowledge foundation led them

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to adopt the project of the well-known arithmetization of mathematics. Numbers should serve to better distinguish between things (Dedekind). The Kantian philosopher Gottfried Martin (1901-1972) has related the difference between Kant and Bolzano to two sides of the development of modern mathematics:

> One can characterize the difference between Kant and Bolzano meaning that for Kant axiomatization, and that for the Bolzano arithmetization has been the ultimate goal. … By the keywords arithmetization and axiomatization the viewpoints are given for a specific assessment of the researchers involved in these investigations. These viewpoints also make understandable, Hilbert`s appreciation of Kant, on the one hand, and Couturat`s, on the other (MARTIN, 1956, p.103).

The central concept of the axiomatic view is *<u>Structure</u>*, the central concept of arithmetization is *<u>Set</u>*. If we represent the complementarity of meaning and reference in terms of Shannon's information theory, we might be justified to say that structure represents *meaning*, while *information* belongs to the notion of sets. Or, rephrased somewhat differently, theories represent meaning, while their applications provide information (see also part **VI.**). Shannon himself addressed this complementarity as follows:

> The concept of information developed in this theory at first seems disappointing and bizarre —disappointing because it has nothing to do with meaning, and bizarre because it deals not with a single message but rather with the statistical character of a whole ensemble of messages, …. The concept of information applies not to the individual messages (as the concept of meaning would), but rather to the situation as a whole (SHANNON/WEAVER, 1964, p.27).

On the other hand, if we identified information with meaning and reduction to what is already known, as is often done, it would imply that nothing new can appear and be known. Set theory and Category theory represent the two fundamental orientations of mathematical activity. We find this split in algebra since early modern times. In fact, there existed until the end of the 18th century two substantially different, but complementary concepts of algebra. "One of these considered algebra to be the science of equations and of their solutions, the other a science of quantities in general" (NOVY, 1973, p. 16). Algebra showed this double character until it passed it on to logic (HEIJENOORT, 1967).

The process of arithmetization led to the rejection of an axiomatic foundation of the number concept, in particular and to the emphasis on cardinal number, i.e. on set theory. Besides set-theoretical foundation became understood as the appropriate form of Platonism in mathematics. The combination of logic and set theory should allow the principle of non-contradiction to be introduced into all of pure mathematics and thus make it completely independent of applications.

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The complementarity of concept and object, or of the meaning and reference of the representations and signs became excluded such that it was not even possible to look at objects that were completely unknown conceptually. The notion of existence became conceived as a second order predicate, that is, as a predicate applied to concepts, rather than objects. McGinn illustrates this view of existence as follows:

"When you think that tigers exist you do not think of certain feline objects that each has the property of existence; rather, you think, of the property of *tigerhood*, that it has instances" (MCGINN, 2000, p. 18).

In this manner, existence pure and simple is ruled out, mathematization as an explorative activity that confronts the yet unknown and uncategorized becomes ignored or excluded (LENHARD/OTTE, 2018). An equation $A = B$ holds, and thereby it differs from the equation $A = A$, besides the identical, that is indicated by the equals sign, something different as well, suggested by the use of the different symbols A and B. According to where one places the identity and the difference, one can see such an equation in different ways. One can conceive of A and B as indicating different objects and then say that the equation designates an equal aspect or an identical property of the different objects A and B. However, one can also conceive A and B as different properties or representations of the same object. For instance, in Grassmann's calculus of extensions. A+B = M are two different representations of the midpoint M of the line segment AB = BA between the different points A and B (OTTE, 1989, p28f).

I.

A "new light" (Kant) must have flashed on the mind of people like Thales, when they perceived that the relation between the length of a flagpole and the length of its shadow enables one to calculate the height of the pyramid, given the length of its shadow. "For he found that it was not sufficient to meditate on the figure as it lay before his eyes, …. and thus, endeavor to get at knowledge of its properties, but that it was necessary to produce these properties, as it were, by a positive a priori construction" (KANT, CpR, Introduction, 1787).

Kant understands that by looking at the diagram, rather than by stumbling around within the confusing host of empirical reality by manipulations of the diagram the result is achieved. This is his doctrine that a mental synthesis precedes every analysis. What really happens is that, although the result is not derived from looking at reality as such,

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the construction of the diagram already requires a preliminary analysis. Kant's ignorance of the role of the object is due to his erroneous conception of space. Kant says:

> If the presentation of space were a concept acquired *a posteriori*, drawn from general outer experience, then the first principles for determining [things] in mathematics would be nothing but perceptions. Hence, they would have all the contingency that perception has; and it would then precisely not be necessary for there to be only one straight line between two points, but this would be something that experience always teaches us (KANT, CpR, A25).

And in fact, if we try and draw a diagram, this truth of phenomenal geometry – that there is only one straight line between two points - becomes immediately obvious. We could then ask for its foundation. For example, why did Kant classify the proposition: *Bodies are extended*, as being analytic, that is conceptual, while considering: *Bodies are heavy*, as synthetic, that is, as depending on objective activity and experience? (KANT CpR, B11/12)? The answer again results from Kant's notion of space.

The logicism of Bolzano and Frege replaced the intuition of space by the conception of logic as a universal language. An example: Wittgenstein's famous *Tractatus* begins like this: "*The world is the totality of facts, not of things*" (*Tractatus*, 1.1). Wittgenstein concluded that all necessity is linguistic necessity. However, his friend Frank Ramsey pointed out to him, that the impossibility of a particle being in two different places at the same time, expresses a feature of the world, rather than of language.

Leibniz had in fact, in his exchange of letters with Bishop Clarke (resp. Newton), claimed that space is relative or relational rather than absolute. Leibniz's argument can be rephrased as follows: Suppose that space would be absolute. Since every region of space would be indiscernible from any other and spatial relations would be construed as extrinsic, it would be possible for two substances to be indiscernible, yet distinct in virtue of being in different locations. But this is absurd, Leibniz argues, because it violates the principle of the *Identity of Indiscernibles*. This principle is fundamental to Leibniz because his rationalism was based on the belief that there must be a substantial reason for everything in the world.

Kant, who had been attached to Leibniz's relational theory of space in his earlier years, turned away from it, because he understood that mere conceptual descriptions cannot capture all there is to spatial relations. Kant used the example of the difference between right hand and left hand, resp. of the two gloves to demonstrate this. If you take a right-handed glove and a left-handed glove, all the internal relations between the various parts of the two gloves are exactly the same, but they represent different *orientations* in space.

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"This is a convincing proof that absolute space is independent of the existence of all matter and has its own reason independently from the possibility of the latter" (KANT, Werke, vol. II., p.994). Kant emphasizes that the idea of space should not be seen as a mere concept, because the possibility of external perception presupposes the concept of space, it follows that space is a pure *a priori* intuition. So, we must be able to intuitively distinguish two orientations in space. Descartes was right, says Kant (and Hegel agrees), in distinguishing between *res extensa* as one of the two substances described in Cartesian ontology, alongside *res cogitans*.

For example, an essential feature of measurement is the difference between the ‘determination’ of an object by direct indication and the determination of the same object by some conceptual means resp. by its relations to other bodies. In the end the latter is only possible relative to objects which must be indicated directly. But this again implies, according to Kant, that there must a kind of spatial intuition been given with the directions, up and down or left and right. This means that we ourselves must have a body, besides being creatures of the mind.

This insight produced the idea of Cartesian coordinates and of analytical geometry. The origin of the coordinate system we use is attached right to our own forehead. This implies that the objects as represented in analytical geometry must be invariant with respect to coordinate transformations. And these transformations are again represented in terms of coordinates (matrices). While the painters formulated the problem of perspective as a relation between the picture and reality. Descartes’ colleague, Desargues (1591-1661) formulated it as a problem of the relation between two pictures and he thus led mathematics on the road towards projective geometry. In the "perspective" geometry of Desargues a circle and an ellipse are considered to be the same mathematical object, for example, since a circle can become an ellipse when the point of view changes. After the preparations of the previous centuries, mathematicians began to classify the various geometrical theories according to the type of transformations that characterize them (*Erlanger Programm* of Felix Klein from 1872). The mathematician studies the properties of geometric objects that remain invariant with respect to certain transformations.

Every activity operates on objects and the objects are always the objects of some activity. Hegel is right therefore when claiming that there cannot be knowledge without an object. He comments on Kant as follows:

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“Kant places the matter somewhat in this fashion: there are things-in-themselves outside, but devoid of time and space; consciousness now comes, and it has time and space available beforehand as the possibility of experience, just as in order to eat it has mouth and teeth, as conditions necessary for eating. The things which are eaten have not the mouth and teeth, and as eating is brought to bear on things, so space and time are also brought to bear on them; just as things are placed in the mouth and between the teeth, they are placed in space and time” (HEGEL, *Werke* 20, p.333-343).

The idea of space was essential to the new sciences of Descartes and Newton. Kant acknowledged that, but he considered space as subjective, as an instrument of knowledge, rather than as its objective condition. If a proof by diagrammatic reasoning is meant to tell us anything about space, then Kant’s derivation should actually be considered a *thought experiment*. From a thought experiment one can learn something about one's own conceptual apparatus **and** about the objective context.

Let us present the following proof of the angle-sum theorem for triangles. Suppose we pass along the periphery of a triangle. By how many degrees we have turned after having reached our starting position again? Simple answer: 360 degrees, because our input direction coincides with the end-position. This response, although intuitively convincing, is based on the assumption that it amounts to the very same thing, to turn around on the spot to a full angle of 360 degrees, on the one hand, or alternatively, do the same thing by travelling along a closed line, the periphery of an arbitrarily large triangle, for example, on the other hand.

One case, however, is based on local characteristics of space, the other is not, at least not if the triangle may be assumed as arbitrarily large! For arbitrary triangles our conclusion is only valid in the Euclidean plane, but is invalid on the surface of the sphere, for example. Spherical geometry is a generalization from Euclidean geometry. The latter is a limit case of spherical geometry when the curvature radius K goes to infinity.

The interesting thing, is that our activity does establish a bridge between subject and object of knowledge, connecting both without denying their difference or distinction. The concept of the epistemic subject and its activity does not only transform or investigate its object, she becomes changed herself in her views and believes. We experience the constraints of the spatial context, so to say, *from the inside*, rather than from the God’s eyes perspective. Kant had said that David Hume has awakened him from his “dogmatic slumber”. Hume claimed that all our knowledge is determined by experience, custom and tradition. “When I see a billiard ball moving towards another, my mind is immediately

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carried by habit to the usual effect and anticipates my sight by conceiving the second ball in motion".

And when Hume had said "the necessity of law is something that exists in the Mind, not in objects, Kant understood this phrase as meaning that Nature must conform to laws, because otherwise the Mind could not understand Nature. We might conclude that all our knowledge is based on evolution and experience and is therefore contextual knowledge. The laws of Nature are laws within the range of the earth and its gravitational field. The laws of geometry are valid as part of the nature of space etc. etc. The first to see clearly the consequences was Hegel. The mathematical evolution stimulated in this way is to be conceived as a process of *assimilation* and *accommodation*. The great anthropologist Gregory Bateson once said:

> The evolution of the horse from *Eohippus* was not a one-sided adjustment to life on grassy plains. Surely the grassy plains themselves were evolved *pari passu* with the evolution of the teeth and hooves of the horses and other ungulates. It is the *context* which evolves! (BATESON, 1972, p. 155).

Evolution of any kind is based on the twofold process of adaptation of the activity or behavior to the context and the assimilation of the context to the activity. One must suit the other and vice versa. So, you can say that knowledge is an activity that is driven in its development by the interplay of *assimilation* and *accommodation*. The first to see this clearly was Hegel. Hegel expected the mental or spiritual universe and the natural world to show similar characteristics.

II.

Having recognized objective activity and technical equipment as foundations of science made Lavoisier the *Newton of chemistry*. And nearly 20 years after the publication of his *Critique of Pure Reason* Kant already an old man, comes to recognize Lavoisier's chemistry as a science. In his *Anthropology* Kant puts Lavoisier, together with Archimedes and Newton, on the same level of excellence and genius: "What amount of knowledge…. would now lie in store, if an Archimedes, a Newton, or a Lavoisier had with their industry and talent been favored with a lifetime lasting through a century" (KANT; WERKE, vol. XII, p.679).

Newton and Lavoisier came together by investigating the relevance of matter and weight, because the "gradual assimilation of Newton's gravitational theory led chemists to insist that gain in weight must mean gain in quantity of matter" (Kuhn 2012, p.71).

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Such beliefs did in the end overthrow the till then dominating phlogiston theory. Lavoisier writes:

> I have been obliged to depart from the usual order of courses … which always assume the first principles of science as known and begin by treating the elements of matter and by explaining the tables of affinities without considering that in so doing they must bring the principal phenomena of chemistry into view at the very outset: they make use of terms which have not been defined and suppose the science to be understood at the beginning. … But all that can be said upon the number and nature of the elements is confined to discussions of a metaphysical matter. …. But if we apply the term element, or principle of bodies to express our idea of the last point which analysis is capable of reaching, we must admit as elements all the substances into which we are capable to reduce bodies by decomposition (LAVOISIER, A., 1965, p.XIX-XXIV).

These words have become famous. For instance, John Dalton (1766-1844), the man who "determined the broad strategy of 19th century chemistry and to some extent physics", wrote in 1810:

"By elementary principles, or simple bodies, we mean such as have not been decomposed. … we do not know that any one of the bodies denominated elementary, is absolutely indecomposable; but it ought to be called simple till it can be analyzed" (CARDWELL, 1968, p.21).

Getting rid of meanings and drawing distinctions that have immediate bearing on our activity is the first condition of research. And this provides mathematics and technology with a privileged role in science. The reference to the physical weight, the consideration of which Lavoisier led to his theory of combustion, points to a general characteristic of research. We usually do not know the nature of things. We do not know their genus, nor their specificities, or their essence and context. We still have to be able to see whether two data structures or points or two objects are the same or are different. In other words, research is made possible by the fact that sense and reference are (relatively) of signs or representations are independent from each other.

The chemistry of Georg Stahl (1660-1734), which dominated until the middle of the 18th century, admitted the existence of indivisible particles, with highly individualized characteristics, but fought against the idea of uniform matter. The atoms should determine the properties of the bodies by their quality and by their individuality. Stahl also thinks that it would be in vain to want to determine the properties of the bodies from the supposed shapes of their particles. Lavoisier, on the other hand, realized that the choice of a scale allows us to substitute for the study of the various intensities of a quality the consideration of numbers, subject to the rules of arithmetic. And since he was a rich

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man, he commissioned the best craftsmen in Paris to produce increasingly sophisticated and expensive gas scales.

> When in the early 1780s Lavoisier and Laplace invented the device that they called a machine for measuring heat, but that soon became the calorimeter, they designed it as an analogue of that epitome of simple machines, the balance. … Despite their collaboration, however, Lavoisier and Laplace recognized somewhat different balances in the calorimeter. With his primary interest in chemistry, Lavoisier saw a balance of chemical substances. … Laplace saw a balance of forces. ……The calorimeter mediated between theories and things. It exchanges theoretical entities for concrete realities (WISE, 2010, p. 208-213).

Lavoisier's *Traité Elémentaire de Chimie,* (1789) contained a clear statement of the <u>Law of Conservation of Mass</u>, and thereby overthrew the theory of phlogiston. His list of substances, however, also included *caloric,* which he more or less believed to be a material substance. And the theory that heat consisted of a fluid (called caloric), which could be transferred from one body to another, but not "created" or "destroyed" was later replaced by the <u>Law of Conservation of Energy</u>, the most important discovery of the second scientific revolution, which was the work of Robert Mayer (1814-1878), Joule (1818-1889) and others, after Sadi Carnot (1796-1832) had paved the ground through his endeavors to understand and improve the steam engine.

Often it was at first not even really understood which things should be measured or compared. This lets us better understand Thomas Kuhn's suggestion that for the *Baconian Sciences* (like Chemistry or Heat and Electricity) there occurred a second *Scientific Revolution* only after Lavoisier, that is, between 1800 and 1850 when many of them became for the first time really mathematized (KUHN, 1979, p.220).

III.

We saw that Hegel criticized Kant's rigid duality of concept and object, of reasoning and intuition, or of quality and being. Kant divides too much between the faculty of knowledge and the object of knowledge. For example, Kant had said in his refutation of the classic proofs of God:

> Being is obviously not a real predicate, it is not a concept of anything that can be added to the concept of a thing. …. A hundred actual thalers do not contain the least more than a hundred possible thalers. For, the possible thalers signify the concept and the actual thalers signify the object and the positing thereof in itself; hence if the object contained more than the concept, then my concept would not express the entire object and thus would also not be the concept commensurate with this object. …. In the state of my assets, however, there is more in the case of a hundred actual thalers than in the case of the mere concept of them (i.e., their mere possibility).

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But that is exactly what Hegel always disliked about Kant's concept of knowledge. Kant seems always to describe the plan of an action, but never comes to the proper action. Concept and object are to be differentiated in a respective moment of the cognitive process, but they play completely symmetrical roles in the overall development of knowledge (OTTE, 1994, p.276ff). In his comments on the above quotation from Kant, Hegel says:

> According to Kant the concept cannot be used to infer being, because being is not part of the concept ... There is no means to recognize the existence of objects of pure thought.... That is to say, that the synthesis of concept and object as a concept *does not happen in Kant's philosophy* ... We have the idea and not the being, the separation of the two is maintained. ... But every activity wants to turn an idea (i.e. something subjective) into something objective (Hegel, Werke 20, p.360f).

Motivation to accomplish something is the real object of activity. And it is, in fact, initially only an idea and is not yet realized objectively. The real motive might not even be consciously present to the mind. But it may probably be realized through the efforts of activity. Hegel's parable of *Master and Servant* tells us how this happens. Hegel conceives of knowledge as an objective activity. This activity is not simply understood as a process, but as a system in development. Assimilation and accommodation, or concept and object, play complementary roles in this developmental process.

Understanding a concept actually means being able to use it. However, if you understand this use or application exclusively from a subjective perspective, - like Kant did - the concept disappears in its function and becomes a mere tool that is not developed, but only worn out. And the object of activity disappears too. In order for the motive of the activity to be realized, actions must be carried out that are not directly related to the motive or object of the activity. Every activity is a system of actions and interactions. For example, to get a book from the university library, I have to do a variety of actions: getting into the car, integrating into the traffic. Upon arrival look for a parking space, enter the university building. Go to the appropriate floor, consult the library catalog ... etc. etc. Each of the activities mentioned can be broken down into parts again.

A. N. Leontiev, perhaps the most important representative of classical psychological activity theory, explains the situation using the example of the hunt:

> The activity of drover or beater at a hunt…. may have been motivated by the need for food and clothing that the captured animal provides him with: What is his activity aimed at, however? The aim is to startle and steer up the animal and to drive it to the hunters. … This completes his work…. Of course, this specific activity of the beater does not yet satisfy his need for food or clothing in itself. The goal to which an activity is directed does not coincide with the motive of its activity ... We call those processes whose aims and motives do

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> not coincide as *actions*, conceiving of the system of actions as the system of activity (LEONTJEW, 1971, p. 168).

A system can constitute itself as a particular system only in difference and interaction with the environment and by the specific functions it assumes within a larger system. Each system develops for the realization of certain functions and it is a subsystem in a more comprehensive context. We must pay as much, or even more, attention to the functional perspective than to the structural one. We seem therefore to run into paradox when trying to generally describe the notion of the systems approach: "Any given system can be adequately described provided it is regarded as an element of a larger system. The problem of presenting a given system as an element of a larger system can only be solved if this system is described as a system" (BLAUBERG *et al*., 1977, p. 270)

The systems approach requires therefore a kind of evolutionary perspective and its paradoxes have to be interpreted as expressing the contradictoriness of a process evolving in time.

The most important prerequisite for learning and knowing is the possibility of simultaneously experiencing a body of knowledge, as well as its development or application. Strictly speaking, this possibility is provided by social cooperation only. Hegel's statement about *Master and Servant* belongs into this category. Just as little as the Master could achieve anything without the work of the servants and could not even guide them with respect to the details of their work, as little could the servant, confronted with the resistance of the material and the details of his task tell his hands or feet exactly what they have to do. The hands think by themselves.

And during the *Industrial Revolution* the hands became substituted by working mechanisms and machinery. Marx made the general diagnosis that it was the machine tool and not the drive (such as the steam engine) that led to the *Industrial Revolution* during the 18th/19th centuries (MARX, 1966, 392f.). *A machine to spin without hands*. "This was the specification of the Jacquard loom in the patent document of John Wyatt (1700–1766) of 1735" (ESSINGER, 2004, p.37).

Drives by water and wind have been around for a long time. What was important was the technology that allowed them to be employed. When the crankshaft reached Amsterdam during the 17th century and revolutionized the production of planks and boards for building ships, because human work could now be substituted by water - which made the conversion of log timber into planks 30 times faster than before, - this transformed Holland into a global economic player together with Britain, substituting

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Spain and Portugal as the dominant sea powers and commercial nations. The immense profits achieved by this technological change provided motivation for a whole nation of shopkeepers, merchants and businessmen.

The attempts of the alchemists to make gold are instructive, because they actually show that an exclusive orientation or even domination by the motive leads to nothing. The alchemists could not bring philosophy and technology together. “That which lives according to reason, lives against the spirit”, said Paracelsus (1443-1541). However, it is also worth noting that without the hope of making gold, alchemists like, Johann Friedrich Böttger (1682-1719) would not have discovered the secret of the creation of hard-paste porcelain in Meissen. Possibility and impossibility are relative terms and must be understood in relation to a system of means and objects. Alchemy only turned into chemistry the moment Lavoisier recognized this connection. For example, if you want to adopt the phlogiston theory of combustion you have to admit - so Lavoisier's argument - that there is matter with negative weight. No engineer or craftsman could live and work in such a world.

The complementarity of motive and action, could be illustrated by the *paradox of mathematical proof*. This paradox can be formulated as follows: On the one hand, the proof can only prove something insofar as the knowledge has a fixed tautological structure and the proof ultimately consists of stringing together immediate logical identities. On the other hand, the proof reduces the new knowledge to the knowledge contained in the premises from the beginning and it is not clear how *new* knowledge can arise in this process. But the paradox dissolves when you consider what the formal system tells you about itself and about its limitations. Against claims that Gödels incompleteness theorem shows that man thinks intuitively, for example, and that humans and computer could be distinguished in this way, Judson Webb has argued as follows:

> The incompleteness theorem shows that as soon as we have finished any specification of a formalism for arithmetic we can, by reflecting on that formalism, discover a new truths which not only could not have been discovered working in that formalism, but - and this is the point that is usually overlooked - which presumably could not have been discovered independently of working with that formalism. The very meaning of the incompleteness of a formalism is that it can be effectively used to discover new truths inaccessible to its proof-mechanism, but these new truths were presumably undiscoverable by any other method (WEBB, 1980, p. 126-127).

On has to draw a distinction to be able to realize why one has to generalize and to pass on into new territory. And one might realize that this procedure – drawing

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distinctions – is the fundamental step of any research activity. We have mentioned it already above when dealing with Lavoisier's work.

IV.

Hegel understands thinking as conceptual thinking and discusses logic or knowledge in terms of language and subject predicate logic. The central theme of Hegel's logic is the concept and thus the linguistic meaning analysis. Hegel regarded "language as the Being of Spirit" (*Phenomenology of Spirit,* Preface).

Language is an incredibly charming, complicated and powerful instrument. People quickly get lost within its labyrinths, puzzles and paradoxes. The eminent Romanist Karl Vossler once said during a speech to students in Munich – very much in terms of Hegelian philosophy: "The true artists of language remain aware of the metaphorical nature of all of their words. They always correct and supplement one metaphor with the other, they let the words contradict each other and only pay attention to the unity and certainty of thought" (see also: DELEUZE, 1990).

People can't even say spontaneously how the following sentences differ:

This rose is beautiful

The raven is black

Paulo Prado is the author of Retrato do Brasil

Even logicians used all the time until the beginning of the 20th century to discover the differences:

1) A is an element of B

2) A is a subset of B;

3) A = B

The important distinction between 1) and 2), that is, between inclusion and membership is normally given to Peano, who employed it in his axiomatic presentation of the natural numbers. However, to people who know the laws of mechanics language is able to produce very subtle nuances of meaning. Look at the following:

The ball was rolling along the grass.

The ball kept on rolling along the grass.

The rather subtle differences of wording in the two sentences suggests all the differences between an Aristotelian conception of mechanical motion and a Newtonian

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and modern one. That is, “the second sentence makes us think of an agent exerting force to overcome resistance or overpower some other force” (PINKER, 1997, p.354).

No wonder that Hegel’s comments on mathematics do not go beyond some mathematically irrelevant subtleties, contrary to the eulogies of the philosophers (STEKELER-WEITHOFER, 1992, p.214ff). Or they appear as somewhat strange and eruptive exclamations. For example, Hegel comments on the traditional Euclidean proof of the theorem of Pythagoras as follows:

> The *essentiality* of the proof in the case of mathematical cognition does not yet have the significance and the nature of being a moment in the result itself; rather, in the result, the proof is over and done with and has vanished. As a result, the theorem is arguably *one that is seen to be true*. However, this added circumstance has nothing to do with its content, but only with its relation to the subject. ….. The nature of a right-angled triangle does not divide itself up in the manner exhibited in the mathematical construction which is necessary for the proof of the proposition….. The necessity does not arise from the conceptual nature of the theorem: it is imposed; and the injunction to draw just these lines, an infinite number of others being equally possible, is blindly acquiesced in, without our knowing anything further, except that, as we fondly believe, this will serve our purpose in producing the proof (Hegel, 1952, p.35/36).

At first glance, Hegel's view of mathematics resembled the attitude of the man who wants to hike from Munich to Milano and complains that the signposts through the mountains were missing. According to Hegel, mathematics falls completely out of the unity of thinking and being that philosophy postulates and pure chance seems to play an undeniable role in its development. Mathematics is not just concept evolution. It does not proceed by empathizing into the meanings of concepts. It is no hermeneutic science, based on interpretations of the vague and constantly oscillating meanings of words. In a similar vein, Ernst Cassirer said that mathematical cognition sets in “precisely at the point where the idea breaks through the cloak of language … to transcend into a principally different symbolic form” (CASSIRER, 1977, p.396).

And Peirce agrees too (PEIRCE, CP 5.147-148), but he feels some sympathy with Hegel as a philosopher:

> The truth is that pragmaticism is closely allied to the Hegelian absolute idealism, from which, however, it is sundered by its vigorous denial that the third category …. suffices to make the world, or is even so much as self-sufficient. Had Hegel, instead of regarding the first two stages with his smile of contempt, held on to them as independent or distinct elements of the triune Reality, pragmatists might have looked up to him as the great vindicator of their truth (PEIRCE, CP 5.436).

Conceptual thinking and metaphysical concepts actually don't play a role in mathematics and the natural sciences. Cardinal Bellarmino (1542-1621), Grand Inquisitor

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and Galileo's principal adversary, in 1615 notified Galileo of a forthcoming decree of the Church, condemning the Copernican doctrine of heliocentrism and ordered him to abandon it as an explanation of the world. He argued that mathematicians always used to speak hypothetically or "*ex suppositione"* only: "First, I say it seems to me that … Signor Galileo acts prudently when he contents himself with speaking hypothetically and not absolutely, … Such a manner of speaking suffices for a mathematician" (Bellarmino, Letter to Father Foscarini of April 1615).

Galileo agreed and disagreed, attributing to religion, resp. science and technology the exactly opposite roles or functions than Bellarmino. In his "Assayer" (Il Saggiatore) of 1623, Galileo compared God's Word in the *Bible*, which is adapted to the frame and imagination of the people, on the one hand, and the *Great Book of Nature*, on the other hand, which presents the realities of Nature objectively as they are and without regard of human interpreters and their desires or preconceptions. Galileo made the point quite clear against the Jesuit Sarsi:

> Possibly he (Sarsi) thinks that philosophy is a book of fiction by some writer, like the Iliad or Orlando Furioso, productions in which the least important thing is whether what is written there is true. Well, Sarsi, that is not how matters stand. Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics, and its characters are triangles, circles figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth.

Pierre Duhem (1861-1916) seems to have been one of the first modern scientists to see the problems involved here and he followed Bellarmino's suggestion. For him all hypotheses based on images are transitory and only relations of an algebraic nature established by theory can stand imperturbably. With respect to the relations between science and metaphysics or religion Pierre Duhem, a convinced and devoted Catholic and a great scientist said:

> If theoretical physics is subordinated to metaphysics, the divisions separating the diverse metaphysical systems will extend into the domain of physics. A physical theory reputed to be satisfactory by the sectarians of one metaphysical school will be rejected by the partisans of another school. … But a physical theory is not an explanation…. It is a system of mathematical propositions deduced from a small number of principles (DUHEM, 1991, p.10f, p.19)

And the well-known German philosopher Hans Blumenberg (1920-1996) claims that science becomes only possible by foregoing a worldview. The excesses of the Nazi era and the struggles against Communism certainly played a role in this (Zill, 2020, p.261). There are in fact many scientists and researchers thinking this way. The opportunity for the autonomy of reason consists in the very fact that nature does not have

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the meaning of a text addressed to man. Blumenberg describes the *Copernican Revolution* accordingly:

> To translate the astronomical concept, according to which stars are lawfully moving light points in the heavens, into the language of the theology of creation in such a way as to respond to the question of the utility and the task of God's heavenly bodies, by saying movement and lighting are their activities, means precisely the liberation of the astronomic object, both from an immediate teleology and from the assumption that this huge expense contained some secret message discernible for man. The opportunity for the autonomy of reason consists in the very fact that nature does not have the meaning of a text addressed to man (BLUMENBERG, 1975, p.49).

But there is also the opposite tendency, which interprets the separation of religion and philosophy, on the one hand, and technology and science, on the other, as a crisis in European science. Edmund Husserl is a famous example in case.

V.

We assume that all thinking takes place in terms of signs and symbolic representations and that the complementarity of meaning and reference of the signs is of fundamental importance. Hegel speaks of the unity and contrast of concept and object, as we have seen. But the semiotic terminology helps a lot to make things clearer. And Peirce found that there are three kinds of signs, that are all indispensable in reasoning.

> The first is the diagrammatic sign or icon, which exhibits a similarity or analogy to the subject of discourse; the second is the index, which like a pronoun demonstrative or relative, forces the attention to the particular object intended without describing it; the third or symbol ... signifies its object by means of an association of ideas or habitual connection between the name and the character signified (PEIRCE, CP 1.370).

The icon provides the qualities of its object, but does not contain any existence claim with respect to the latter. The index, in contrast, is just an existence claim without providing any characteristics; it is in general physically connected with its object. The symbol on the other side becomes established by social convention. This classification of signs takes the sign-object relation as its starting point.

Catherine Elgin objects to Peirce's classification. She does not believe that there are any signs, "that are simpler and more easily grasped than conventional symbols" (ELGIN, 1997, 146). Differently from symbols the other signs too, she says, seem to bear the same type of relationship to their objects

> whether they were interpreted as doing so or not. …. The difficulty is that resemblances and natural correlations are ubiquitous. Every two entities bear some likeness to each other, and some correspondence in fact. Yet we do not

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> consider every object a sign, much less an icon or index of every other … . Something is an icon or index only if it functions as such. … But being taken to signify requires an interpretant. So icons and indices, like conventional signs, are symbols. … Icon, index and symbol threaten to collapse into an undifferentiated heap (ELGIN, 1997, p.143).

But this is a consequence of philosophical nominalism, that is, of the view that the subject determines the nature of a sign and thus becomes the source of signification, rather than the object. Peirce called the proper significate outcome or effect of a sign its *interpretant*. This is commonly called the *meaning* of the sign, and it is obviously a further sign determined by the original sign itself. Elgin seems to assume in contrast that the interpretant is a person, rather than a sign. "The interpretant is that which guarantees the validity of the sign even in the absence of the interpreter. It is a sign that translates, explains, makes clear, analyses or substitutes for the sign which gave rise to it" (CHANDLER, 2017, p.35).

From a psychological point of view – Elgin's view – we might say that a concept has a meaning. And this meaning is as much influenced by what a person had experienced and knows as well as, by any new application intended. Every new experience changes the understanding of the conception and every new understanding leads to different applications and experiences. Interpretation and understanding always seems a mix of the intellectual or artistic experience of the receiver and of the result of new experiences from a concepts or instruments applications. A musician, - once arguing against those who held a too narrow conception of music - said: "Since whatever music seems to communicate to somebody is … touched off by his auditory experience, … it may mean as much or as little as life itself" (OTTE, 2014, p.149).

That is, it means whatever it may mean. Identifying a text with communicative function leads to a psychologistic theory of meaning. That is, insisting on the question *what did the author really mean* has no more merits to it than the idea that the reader, rather than the author is the source of meaning. Neither author nor reader can give notice of their meaning experiences, differently than by producing a text. Texts relate to texts, signs relate to signs. Not even the author of a text can reproduce its original meaning, because nothing can bring back the original meaning experience.

And if you bump with your head against something hard in the dark, you become convinced that there is something resisting your head, even though you might not know what it is, a bar or a warning from God and destiny. The bump against the head is an *index* of something hard, in the very same manner as fever is an index of illness, usually

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interpreted as a symptom of inflammation, although one might not be able to really diagnose the disease. The fever is, semiotically speaking, an *indexical sign* of the disease. Fever indicates the fact of sickness, but does not describe its character. One might certainly use its symptoms, *heat, redness, trembling, hectic activity* etc. and refer to them in terms of metaphorical speech as “fever” of some kind and as a consequence the word “fever” is applied in quite a number of different contexts. But this does not justify to call fever an icon or a symbol and to conclude that “icons and indices, like conventional symbols, are symbols” (ELGIN,1997, p.139).

But the most important aspect of the distinction between indices and icons, is the question of how communication and the reference to the object can be coordinated. Leibniz had two projects to do this, but he was unable to carry them out. In the 19th century they led to two opposing ideas of logic and mathematics (Lenhard / Otte 2018). And because all thinking occurs in terms of signs, to interpret something just means to represent it. The essence of something is nothing, but the essence of a representation of that thing. and the dynamical interpretant. We can ask neither for the ultimate referent, nor for the definite meaning of a sign.

The concept of complementarity is not just mean duality of icon and index, but means their connection. Charles Peirce's calls that *Thirdness*. Peirce writes: “*Thirdness* is the triadic relation existing between a sign, its object and the interpreting thought, … considered as constituting the mode of being of a sign” (PEIRCE, CP 8.332).

VI.

Mathematicians commonly consider arithmetic and geometry as different with respect to epistemological and ontological status. Frege characterizes, for example, arithmetic as conceptual and analytic, geometry, in contrast, is regarded as synthetic, either because of its constructivity (Euclid) or because it is based on arbitrary principles and definitions. With regard to the difference between arithmetic and geometry, Hao Wang writes:

> We are familiar with the two usual methods of developing mathematics: the genetic or constructive approach customary in the extension of numbers to integers, fractions, real numbers etc.; and the axiomatic method usually adopted for the teaching of elementary geometry…. The application of the axiomatic method in the development of numbers is not natural. Its rather late appearance is evidence (WANG, 1970, p. 69).

This statement is remarkable, as much for what it says as for what it omits. We know that the axiomatization of numbers began about two thousand years after Euclid’s

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axiomatic presentation of geometry with the publication of Hermann Grassmann's small textbook "Lehrbuch der Arithmetik" of 1861. The traditional recursive definitions of addition and multiplication are due to Grassmann:

x+0 = x; x+(y+1) = (x+y)+1;

x*0 = 0; x*(y+1) = (x*y)+x.

In this way addition and multiplication of natural numbers are derived from one single operation: x+1. Grassmann´s exposition essentially corresponds to the characterization "which is customary in present day abstract algebra" (Wang, 1970, p. 70).

Numbers are concepts or functions. And arithmetic begins with *theoremata* like: *The product of two odd numbers is odd*. Or: *If an odd number divides an even number without rest, it also divides half that number without rest*.

These theorems go beyond what can be experienced concretely in intuition (be it pure or empirical), because they state something about infinitely many objects. Actually, they do not state anything at all about objects, but they are analytic sentences, which unfold the meaning of certain concepts. What does it mean to say that *X is an odd number*? It means that there is another number *N* such as that $X = (2N + 1)$ holds. If we have two odd numbers represented in this way before us, and if we multiply these, the said theorem will result automatically by applying the distributive and commutative laws. The proof unfolds the sense or meaning of the number concept.

An axiomatic theory defines concepts, rather than objects. It refers attributively to reality rather than referentially. For example, Peano's axioms do not answer the question "What are numbers, what is the number 1 or 2? Numbers could be anything, even games (Conway-Numbers, Hackenbusch-Games, Chessboard-Computer, geometrical Vectors, etc.). This means that axiomatic statements like A + B = B + A, are to be interpreted as hypothetical judgments which assert something for the case that some numerical symbols are given. These statements do not presuppose the existence of a totality of individual objects, called natural numbers, and telling us something about these objects.

Let us simplify things a little. Let our theory of arithmetic consist of only **one** axiom:

If *a* and *b* are numbers, then $(a+b) = (b+a)$

Suppose two countries. One endorses a discrete world-view and considers numbers to be equivalence classes of sets. 3 is the class of sets with just three elements $$$! Can numbers be such sets?

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Test: ($$) +($$$) = ($$$$$) = ($$$) + ($$)

So sets are numbers! The people of the other country (Aristotle may have lived in such a country) say "No, numbers are translations (classes of vectors). The reader may verify with ease using pencil and paper that vector-addition obeys our axiom such that in this country numbers are vectors!

As a rule, the situation is viewed from the other perspective, that is from the side of two groups of structurally arranged objects whose isomorphic structure has to be ascertained. Let us assume that we have two groups of geometric transformations on the plane. Both consist of exactly two elements. One G1 consists of the identical transformation *I* and a reflection *A* on a given straight line, the other G2 consists of the identity *I* and a rotation *B* by 180 degrees around a fixed point. Then both are defined by the axioms I.I = I, I.A = A.I = A and A.A = I; or: B = B.I = B and B.B = I. This reverse perspective is much simpler and more common, one has already the structure and you should only try to verify isomorphism or difference of two structures. The other direction, which is to detect a structure within a set of objects, is much more difficult.

Let us give a less trivial and historical highly important example of structural thinking. To cope with the algebraic irrationals, like the square root of 2, Euler, in his *Complete Guide to Algebra,* used a mixed notation: $a + b\,(2)^{1/2}$ or if we abbreviate $(2)^{1/2}$ by *t*: a*1* + b*t* for the enlarged set of the algebraic numbers, as he did not see how he could deal with the irrational numbers otherwise. To anybody having the notion of *algebraic structure* at hand this suggests the analogy with the imaginary numbers, because *1* and *t* are linearly independent vectors over the rationals, in exactly the same way, as *1* and *i* are linearly independent vectors over the reals. The mere semiotic representation created a new ontology. However, the axiomatic idea of a vector space did not yet exist during the 18th century nobody saw and explored that analogy before the 19th century such that a new mathematical disciplines, like algebraic number theory or vector calculus and linear algebra had to wait.

Structuralist trends appeared since the advent of "pure mathematics" at about the turn to the 19th century and structural analogy became a powerful research instrument. The central concept of the axiomatic view is <u>*Structure*</u>, the central concept of arithmetization is <u>*Set*</u>. *Set theory* and *Category theory* represent the two fundamental orientations of mathematical activity. Ernst Cassirer describes the fierce controversy surrounding the foundation of the number concept between two main directions, as representative of which he names Helmholtz, Dedekind and Peano, who axiomatically

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justify the number concept by means of the ordinal number, and on the other hand, Cantor, Frege and Russell. Cassirer writes:

> The first basic view of knowledge from which one can start out is that all knowing has to fulfill a representative function. (…) This conception of the affair contrasts with another, which may be called the functional view of knowledge. (…) For the object is not treated as a given fact [das Gegebene] but as a problem [das Aufgegebene]; it serves as the goal of knowledge, not as its starting point (CASSIRER, 1973, p.66–70).

The point is that the object must be the starting as well as the end point of the cognitive activity. Therefore, it must be an object and an idea, an individual and a species, simultaneously. That was Hegel's problem (and it had been Plato's problem already).

Mathematical development is based on abstraction starting from abstractions and operations, rather than from objects (PEIRCE, CP 4.234, 5.447 and NEM IV, p.49). Hypostatic abstraction is achieved by hypostatizing a predicate, a process or a quality, thereby turning it into an object, capable of being further investigated. We transform, for instance, propositions, like, *honey is sweet*, into, *honey possesses sweetness*. This may sound trivial, although it facilitates such thoughts as that the *sweetness of honey is particularly cloying* or that *the sweetness of honey is something like the sweetness of a honeymoon*; etc. Language appears to be a "flat game" in this respect, when compared to computer science or even to mathematics. The computer scientist E.W. Dijksta writes:

> Compared with the depth of the hierarchy of concepts that are manipulated in programming, traditional mathematics is almost a flat game, mostly played on a few semantic levels, which are moreover are thoroughly familiar. The great depth of the conceptual hierarchy … is one of the reasons why I consider the advent of computers as a sharp discontinuity in our intellectual history (DIJKSTA,1986, p. 49).

But, as Hegel might argue, the mathematician or computer scientist lacks the biologist's or gardener's insight into the fact that the object is also active and is not just dead matter. For example, the sweetness of a certain selection of honey might be more charming than a second different one. Or, one type of parrot could be more beautiful than a different one, etc. This is an everyday experience for the biologist or farmer. The relations between continuity, variation and possibility influence all theories of evolution. Ernst Mayr, sometimes considered the "Darwin of the 20th century", for example, distinguishes between "typological thinking (essentialism)", founded, as he says, by Plato, and "population thinking", which he ascribes to Darwin. As an example of essentialism, he cites concepts like the famous "general triangle" from geometry. Ernst Mayr continues:

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> What we find among living organisms are not constant types, but variable populations … Within a population … every individual is uniquely different from every other individual". In addition, if the differences between individuals become sufficiently large, two species might suddenly break away where there had been just one before. Darwin's "basic insight was that the living world consists not of invariable essences (Platonic classes), but of highly variable populations. And it is the change of populations of organisms that is designated as evolution (MAYR, 2001, chapter 5).

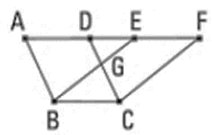
And from a certain perspective – abandoning the set-theoretical foundation of mathematics, one might treat a general triangle as representative of a kind, and not as a collection of determinate triangles, exactly the same as when we say: "An apple is a fruit". For example, the sentence "an equilateral triangle is a general triangle", is true in the context of affine geometry, but is false in Euclidean geometry. It is an idea, a sign which governs and produces its particular expressions. Such a perspective obviously generates new proof ideas and proof procedures (OTTE, 2006, pp147-153).

VII.

Euclid's *Elements* are considered as the first axiomatic presentation of geometry and of mathematics in general. But they are concerned with the possibilities of constructing real geometrical figures and are thus synthetic. In fact, Euclid systematized handling real geometrical objects (Fowler 1987) and judged the geometrical postulates and theorems on the basis of a justified and experience-based certainty.

For example, in the very short argument of §35 (theorem 25) of book I of Euclid's *Elements* the word "equal" occurs more than 10 times, with three different meanings: *congruence*, *equality of area* and *numerical identity*. The theorem reads:

"*The parallelograms which are on the same base and in between the same parallels equal one another"*.



The question arises what Euclid means in using the word "equal"? David Fowler says that the "the idea behind Euclid's use of equality within geometry is one of size not one of shape and his concern is to see if two plane figures are equal in size" (Fowler, 1987, p.13). The basic equality is thus numerical identity.

Euclid's geometry is a geometry of figures, not a geometry of space. And the figures generate one another. The Euclidean point of view is genetic, not ontological. Every theorem fulfills certain functions within the whole structure of the theory. And the

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theory itself is created as a kind of multifunctional problem-solving device. Take the proof of Theorem 44 of Book I.

- To prove it one refers back to Theorem 42, Theorem 29 and Theorem 15 and in addition to Axiom 8 and Postulate 5.
- and to prove Theorem 42 we refer to Theorems 10, 31 and 41.
- and to prove Theorem 41 we refer to Theorems 34 and 37, etc., etc.

Euclid deals with the angle-sum theorem of the triangle in proposition 32 of Book I. The proof of this theorem makes use of theorems 13, 29, and 31, which in turn rely on theorems 11, 13, 15, 23 and 27, and so on, back to the postulates.

This structure is not based on a logical-deductive connection, but it arises from the activity of solving plane geometric problems. On should also remind oneself that the essential postulates of Euclid's *Elements* are presentations of admissible constructions. We read, for example: “Let the following be postulated: *To draw a straight line from any point to any point*”, or: “*a circle can be drawn with any point as its centre and with an arbitrary radius*”, …., etc. etc. Under these conditions, a proof in Euclid’s *Elements* is nothing but the demonstration that if certain operations or constructions are licensed, something can be constructed. One might think of the strange first theorem: “*On a given straight line to construct an equilateral triangle”.* Why call this a theorem, rather than a construction problem?

It is always a matter of solving certain construction problems, which in turn require the solutions of other problems. A theorem of Euclid embedded into the structure of his *Elements* represents a certain function or an element fulfilling a certain function within a wider problem-solving context. Andrey Kolmogorov, one of the greatest mathematicians of the 20th century writes:

> In addition to the theoretical logic which systematizes the proof schemes of the theoretical truths, one can also systematize the solutions of problems, e.g. of geometric construction problems. In analogy to the principle of syllogism, the following principle holds here: if we can reduce the solution of *b* to the solution of *a* and the solution of *c* to the solution of *b*, then we can also reduce the solution of *c* to the solution of *a*. .... Thus, in addition to the theoretical logic, a new calculus of problems is obtained. One does not need any special epistemological or intuitionist assumptions (KOLMOGOROV, 1932, p. 58).

Aristotle explains the geometric evidence as constructive realization of pre-existing possibilities. And he uses the example of the proof of the angle sum theorem in the triangle to emphasize that which might exists as a possibility is actually found by means of real activity. Aristotle hereby testifies that the analogy between Euclid’s constructivism and the logic of problem solving is not unfounded. Aristotle says:

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> Why are the angles of the triangle equal to two right angles? Because the angles about one point are equal to two right angles. If, then, the line parallel to the side had been already drawn upwards, the reason would have been evident to any one as soon as he saw the figure. Why is the angle in a semicircle in all cases a right angle? If three lines are equal - the two which form the base, and the perpendicular from the center - the conclusion is evident at a glance to one who knows the former proposition. Obviously, therefore, the potentially existing constructions are discovered by being brought to actuality; the reason is that the geometer's thinking is an actuality; so that the potency proceeds from an actuality; and therefore it is by making constructions that people come to know them (ARISTOTLE, *Metaphysics* 1051a30).

In 1919 Einstein published an article in the *Times* describing different types of theory:

"There are several kinds of theory in physics (and in mathematics too, our insertion). Most of them are constructive. These attempt to build a picture of complex phenomena out of some relatively simple proposition. …. But in addition to this most weighty group of theories, there is another group consisting of what I call theories of principle. These employ the analytic, not the synthetic method. Their starting point and foundation are empirically observed general properties of phenomena, principles from which mathematical formula are deduced of such a kind that they apply to every case which presents itself. …The merit of constructive theories is their comprehensiveness, adaptability, and clarity; that of the theories of principle, their logical perfection, and the security of their foundation. The theory of relativity is a theory of principle".

The Michelson-Morley experiment, showed that the velocity of light is constant and independent of the relative position and movement of the source.

The original purpose of the famous Michelson-Morley experiment of 1887 was to measure the speed with which the earth travels through the ether. For centuries, from Newton onward, it had been a well entrenched tenet that something called the *ether* pervaded all of what we think of as empty space. The great physicist Lorentz (1853-1928) had hypothesized that the ether itself was stationary. What the experiment revealed was that the method that was expected to enable measurement of the earth's speed through the ether was totally inadequate to that task.

Lorentz, in an effort to save the hypothesis of stationary ether, introduced a number of physical modifications and ad hoc ideas and shifted to a new and more complicated set of formulas in his mathematical physics. Einstein relying exclusively on a set of principles about light and motion soon cut through all this, propounding the special theory of relativity. The story becomes even more interesting after Hermann Minkowski, Einstein's mathematics teacher at the *Federal Polytechnic School* in Zürich,

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proposed – taking a move which he himself characterized as the result of the *boldness of mathematical culture* (Blumenthal 1982, p.60) – to interpret Einstein's new physics as a geometric theory of a four-dimensional metric space having three space-dimensions and one time-dimension. Einstein's theory becomes nothing but the theory of geometrical transformations that leave a certain indefinite quadratic form invariant. This picture is completely analogous to the interpretation of ordinary Euclidean geometry as the theory of invariance with respect to the quadratic form by which the usual metric is defined.

VIII.

Since the turn of the 19th century all theories of mathematics and natural science have become theories of principle, in the sense of Einstein. And Hegel's phenomenology was a philosophy of principle, an attempt to put philosophy in the position of orienting and judging all areas of culture on the basis of some fundamental principles. The idealistic view of Hegel's philosophy today contrasts primarily with the logical empiricism and positivism of analytical philosophy. This is shown in the following statement by the prominent Hegelian philosopher Rüdiger Bubner (NZZ from October 23, 1987): "We do not become the master of technology through technology".

This statement obviously allows an exalted and a realistic interpretation. On the one hand, the metaphors of seeing and feeling and spiritual reflection and rumination dominate: in this case, the sentence is formulated against the background of the tacit assumption that we have sufficient means in philosophical consciousness and reflection to determine technology. The indefinite determines a part of itself, as it were, the technical. The spirit – especially the spirit of God – determines matter, the concept the object. The infinite, says Leibniz, determines the finite. And the infinite is reserved to God's Spirit.

A limited rationality is attributed to technical constructivity and scientific knowledge, a way of knowing that regards the world only functionally and for certain purposes and does not care about the nature of the reality in which it operates. It focuses only on the solvable problems and neglects the fact that every solvable problem is part of an unsolvable problem. Technical activity and pragmatic concern cuts off systematic connections and more distant side effects. In this limited rationality, thinking is reduced to constructing, calculating, measuring or deciding between technical alternatives, without a comprehensive overview of the purposes, goals and consequences. Reduction

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of complexity is obviously always necessary. We cannot take the world as a whole into account. The resistance opposing our efforts would be infinite. In fact, a constructive approach first has to make fewer assumptions about its subject than a descriptive one, and it therefore gains easier access to reality.

We have seen that the creation of chemistry by Lavoisier and others gained from Newtonian mechanics. And the same is true for biology. The science of the inanimate has historically enabled a science of the animate. Only after the classic teleological view of nature had been overcome did a mechanistic science appear, which then made its counterpart – the human world - tangible. The heroes of French *Enlightenment,* like Diderot (1713-1784) or Buffon (1707-1788), not accepting man as the unconditioned source of knowledge, nevertheless placed man in the center of *Natural History*. "Animal species appeared successively in the order of their proximity to man: domestic animals than wild animals" (ROGER, 1997, p.228).

Hegel perceived this course of the history of philosophy and science. Descartes stands at the very beginning of modern European philosophy. Hegel acknowledges this, saying:

> René Descartes is indeed the true beginner of modern philosophy in that it makes thinking a principle. …. The great effect that *Cartesius* had on his age and on the formation of philosophy in general lies in a free and simple, at the same time popular way ... starting from the popular thought itself and from very simple sentences, reducing the content to thinking and material being (res cogitans and res extensa). ..... This is the way of his time. What the French called exact sciences, sciences of a certain cast of mind, began with this period. Philosophy and exact science were not separate; only later did the two seperate (HEGEL, Werke, vol.20, p.123).

It should be noted that every activity affects not only the objects of the activity, but also the subject itself. We already see this today in the resistance to the transformation of the landscape by the new technologies. The technical change is more rapid than culture, costumes and self-images of people could bear or accept. In the 19th century philosophy is not a science anymore - as Hegel says in his inaugural lecture at the University of Berlin - but it is certainly indispensable if people want to gain a realistic view of life. Historically, the idea of *German Idealism* began with Hegel and Hölderlin two friends at the end of the 18th century, students and roommates of the *Tübinger Stift* (Tübingen seminary), an institution for the formation of Protestant priests, public servants and teachers. Hölderlin fled the involved philosophical reasonings of his friend, Hegel, believing that the intricacies and confusions of philosophy have to be resolved in poetical work (SAFRANSKI, 2019, p.305).

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In his note entitled: „*Urteil und Sein*“ Hölderlin had stated: *Ich bin Ich!* (Hölderlin 1965, p.947). *I am I* , or: *I = I*, does not mean dead identity. The *I* is the object (subject of the sentence) as well as the concept (predicate). Hölderlin meant his formula as a poetic principle and had seen the goal of poetry as ultimately self-knowing by unification with the outside world of objects. It is often said that without the valley of the *Neckar* river there would have been no Hölderlin, Germanys greatest poet. And in fact, Hegel himself had conceded (Hegel, 1981, p.49-50) that the *I = I*, can either be interpreted as a subjective identity of subject and object, as a pure intuition, or it can be understood in the sense of a Platonic maxim, according to which man must experience the whole world - to measure himself against all kinds of things - before he can get to know himself.

But art and poetry always also include a distancing from general social life. And poetry always represents a rupture between life and spirit (Geist), which Hegel does not accept. In the Preface to his *Phenomenology of Spirit* Hegel presents an extensive critic of his friend Hölderlin, without mentioning the latter’s name. In art and poetry, he writes, “the Absolute should not be grasped, but felt and looked at, not its concept, but its feeling and intuition should lead the word” (HEGEL, 1952, p.13). Hegel considers this to be a form of ego-centrism that must lead to mental defects or even mental illness (STRASSBERG, 2014, p.217-250).

Goethe, Hölderlin and the Romantics thought differently. Hegel’s friend Goethe explained that a separate discipline called *Philosophy* seemed superfluous to him:

> My friend began, namely, to make me acquainted with the secrets of philosophy. He had studied in Jena and had acutely seized the relations of that doctrine, which he now sought to impart to me. ….. Our most important difference was this, *that I maintained a separate philosophy was not necessary*, as the whole of it was already contained in religion and poetry. This he would by no means allow, but rather tried to prove to me that these must first be founded on philosophy; which I stubbornly denied, and at every step in the progress of our discussions, found arguments for my opinion. For, as in poetry a certain faith in the impossible, and as in religion a like faith in the inscrutable, must have a place, the philosophers appeared to me to be in a very bad position when trying to demonstrate and explain both from their own field of vision. ….. (GOETHE, 1998, p.200).

Rüdiger Bubner argued from a Hegelian position against Goethe’s opinion:

> What Goethe implies with the descriptions of the impossible in which poetry believes, and the unfathomable, in which religion believes, is what Hegel calls the operational concept of the absolute. The Absolute provides an abstract term for that which extends beyond the finite limitations of the world of understanding. … .. Since art cannot stick to anything solid, objectively given to the general understanding, …. it must just dare to do what it does. Similarly, religion has to do with the unfathomable, that is, with the sublime beyond all rational reasons. So only faith helps (BUBNER, 1995, p.179).

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According to Bubner, Hegel describes "the definitely valid way of perceiving the absolute as the *Spirit*". Because everything subjective is indeterminate and accidental the Spirit or Mind must be something subjective as well as objective and must be a moment in the evolution of reasonableness. Hegel seems to have confirmed this view. In the introduction to his lectures on *The Philosophy of Right (or Law)* he said to his students in Berlin:

"Philosophy is an inquisition into the reasonable or rational, and therefore the apprehension of the real and present. ……… *What is rational is real; And what is real is rational.* Upon this conviction …. proceeds the view now under contemplation that the spiritual universe is the natural."

2 Conclusion

What Goethe approved in Hegel was the principle of his intellectual activity, which centered around the mediation of subject, object and human history. It was this interest that also connected Hegel with Peirce. However, what distinguishes Peirce from both Goethe and Hegel is the appreciation of mathematics and the associated insight into the importance of semiotics and symbolism beyond mere language.

The Romantic movement that began around the time of Hegel's birth generally understood itself as a reaction to the *Industrial Revolution* and as a corrective to the mechanization of the world view. But both, Romanticism and philosophical Idealism have consistently noticed that their concern showed, as it were, paradoxical features. Even the two worlds of philosophy vs mathematics and technology cannot be brought together to a final synthesis outside the dynamics of history and evolution. The aspirations of the Romantics and of philosophical Idealism and their discomfort at the age and the search for the middle or a compromise accompanies us to this day.

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